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Threefolds with one apparent double point

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Introduction

Given an irreducible variety X of dimension n in \mathbb{P}^{2n+1} , the number of *apparent double points* of X is the number of secant lines to X passing through a general point of \mathbb{P}^{2n+1} . If this number is 1, then X is called a *variety with one apparent double point*, or OADP variety. Roughly, X is said to be *secant defective* if through a general point of \mathbb{P}^{2n+1} there is no secant line to X . Therefore OADP varieties are considered the simplest non defective varieties. However, their classification is very rich and challenging.

In dimension one, the only OADP variety is the rational normal cubic (see for instance [CR, Proposition 2.2]). The classification in dimension two started with Severi. In [Sev], he stated that the OADP surfaces having at most a finite number of singularities were degree four rational normal scrolls and weak Del Pezzo surfaces. A gap in his arguments was fixed in [Ru1], which considers only the smooth case. More recently, Ciliberto and Russo classified all OADP surfaces, including the Verra Varieties to Severi's list, see [CR].

The classification of smooth OADP threefolds was done by Ciliberto, Mella and Russo, in [CMR]. In the same work, the authors have also shown an important property of a smooth OADP variety X : if x is a general point of X , then the projection $\tau : X \dashrightarrow \mathbb{P}^n$ from the projective tangent space $T_x X$ is birational. This also holds for singular OADP varieties, as remarked in [CR]. The inverse of τ contracts to the point x a hypersurface $V \subset \mathbb{P}^n$, which the authors called *fundamental hypersurface*. They studied the case in which V is a hyperplane, giving a partial classification of OADP varieties for arbitrary dimension n .

The aim of the present work is to study singular OADP threefolds such that the fundamental surface V is not a plane. This will be done through the analysis of linear systems \mathcal{X} in \mathbb{P}^3 that define the inverse of a tangential projection τ of X .

This approach classifies the so-called *Bronowski varieties*, i.e., varieties such that the general tangential projection τ is birational. As explained

above, OADP varieties are Bronowski. The converse is known as the *Bronowski conjecture* and it has been first claimed in [Br]. All examples of Bronowski varieties that appear in our study are also OADP.

Some remarks made in [CR] allow us to simplify the study of Bronowski threefolds. The first remark is that V is a projection of the Veronese quartic surface to \mathbb{P}^3 . The case in which V is a quadric is studied in Chapter 2. In Chapter 3, the cubic case is considered. Finally, in Chapter 4, the case in which V is a quartic surface is analysed.

The second remark is that the linear system \mathcal{X} which defines the map $\tau^{-1} : \mathbb{P}^3 \dashrightarrow X \subset \mathbb{P}^7$ has degree $d \geq 2\delta + 1$, where $\delta = \deg V$. Here, we focus in the cases in which equality holds, that is, we assume the following hypothesis:

- (H) Let X be a normal Bronowski threefold such that the linear system \mathcal{X} defining the inverse of the tangential projection at a general point $x \in X$ has degree $d = 2\delta + 1$, where δ is the degree of the fundamental surface.

The hypothesis on the degree of \mathcal{X} may seem unnatural at a first glance, but it will be explained later that, for instance, it holds in the smooth cases (see page 13).

Other properties of \mathcal{X} explained in [CR] are used in the classification (see Lemma 1.3.2 below).

By understanding the different linear systems \mathcal{X} that can define a map τ^{-1} , we can study the varieties X given by the images of this map. The obvious way to do that is to blow up the base locus of \mathcal{X} . This is simplified by the fact that, in most of the cases, the curves in the base locus of \mathcal{X} are rational curves. Another option is to look for a family of surfaces in X that span projective spaces of low dimension. Then we can use the fact that X is *linearly normal* (see page 16) together with a result on linearly normal varieties from [EH] to prove that it lies on a rational normal scroll.

The singularities of X can be understood with the help of Lemma 1.1.4 and Lemma 1.3.3. In addition, the fact that τ is a projection implies that most of these singular points are mapped to points in \mathbb{P}^3 (more precise statements are given in Lemma 2.2.9, Lemma 3.2.4 and Lemma 4.4.1).

Most of the threefolds considered in this thesis are degenerations of known OADP threefolds. The ones in Chapter 2 are degenerations of the Edge variety of degree seven. In 6 of the 19 degenerations, the fundamental surface is a cone. The varieties in Chapter 3 are degenerations of the degree eight smooth scroll in lines described in Example 1.4.2. In all of them, the fundamental surface is the general cubic scroll with a double line.

All the threefolds that appear in Chapter 4 are singular, so they do not appear in the classification in [CMR]. They have degree nine and all of them have a point of multiplicity four. In the general threefold, the tangent cone at this singular point is a cone over the quartic Veronese surface. The fundamental surface can be any of the three quartic Steiner surfaces, each producing different OADP varieties.

Considering the classification of smooth OADP threefolds presented in [CMR], these examples of degree nine are quite unexpected. This shows that not only smooth OADP varieties are interesting, but also the normal ones.

The main results of this thesis are the following:

In Chapter 2, a full classification of the case in which V is a quadric is given, with a description of the singularities, in Theorem 2.4.1. In the same result there is a description of these varieties as residual intersections of quadrics with a cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$, which allow us to prove that those are OADP varieties. In Table 2.1, the 19 varieties fitting in this case are listed.

In Chapter 3, the number of varieties produced is very high, so an explicit list is not given. This is due to the number of possible degenerations of a curve in the base locus of \mathcal{X} . A hypothesis on the base locus of \mathcal{X} was added, namely that it has pure dimension one. This is due to the fact that we do not study in detail its scheme structure, so there could be unexpected embedded points. The types of singularities of X are explained and a geometric description is given (see Theorem 3.6.2). The fact that these threefolds are OADP is also proved.

In Chapter 4, a similar hypothesis is also made on the base locus of \mathcal{X} . The singularities of X are explained in Proposition 4.4.6. We prove that the general variety is OADP, by describing it as an intersection of divisors in a certain cone.

Chapter 1

Preliminaries

1.1 Notations and basic definitions

Let $X \subset \mathbb{P}^N$ be a *variety*, that is, a reduced projective scheme over \mathbb{C} , and x a point in X . In what follows, $T_x X$ denotes the *embedded projective space* to X at x .

Let $\varepsilon : \text{Bl}_x(X) \rightarrow X$ be the blow of X in x , with exceptional divisor E . If y is a point in E , we say that y is *infinitely near to x* , writing $x \prec y$. When doing blow-ups, I refer in the same way to varieties and their strict transforms, if there is no danger of confusion.

More generally, we say that y is *infinitely near to x of order n* if there is a sequence of blow ups:

$$X_n \xrightarrow{\varepsilon_n} X_{n-1} \xrightarrow{\varepsilon_{n-1}} \cdots \xrightarrow{\varepsilon_2} X_1 \xrightarrow{\varepsilon_1} X_0 = X$$

where ε_i is the blow up at x_{i-1} , such that $\varepsilon_i(x_i) = x_{i-1}$ for $i = 1, \dots, n$, $x = x_0$ and $\varepsilon_n(y) = x_{n-1}$.

If a point $x \in X$ is not infinitely near to any other point in X , we say that x is *proper*.

I will now introduce some notations.

For rational normal scrolls, the same notation of [EH] will be used. So given $0 \leq a_0 \leq \cdots \leq a_d$, $S(a_0, \dots, a_d)$ is the image of the projective bundle $\mathbb{P}(\mathcal{E})$ via $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$, where:

$$\mathcal{E} = \bigoplus_{i=0}^d \mathcal{O}_{\mathbb{P}^1}(a_i)$$

In other words, it is the scroll described by the \mathbb{P}^d 's joining corresponding points of $d+1$ rational normal curves of degrees a_0, \dots, a_d (where the rational

normal curve of degree zero is a point). It has dimension $d + 1$, degree $a_0 + \cdots + a_d$ and spans a space of dimension $a_0 + \cdots + a_d + d$.

For a curve $C \subset \mathbb{P}^3$, the exceptional divisor of its blow up is denoted by E_C and the intersection of the strict transform of a surface (or a linear system) S with E_C is denoted S_C . The same notation is used in the blow up of points.

Moreover, if C is a rational curve and it is the complete intersection of two surfaces $C = S_1 \cap S_2$, then its normal bundle is:

$$N_C = N_{C/\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$$

where $C^2 = a$ in S_1 and $C^2 = b$ in S_2 . Then, blowing up C , it follows that $E_C \cong \mathbb{F}_n$, where $n = |b - a|$, since E_C is isomorphic to $\mathbb{P}(N_C)$.

If C is a curve and $E_C \cong \mathbb{F}_0$, I will denote the divisor classes of E_C as (a, b) , where $(1, 0)$ are the fibers over points of C . If $E_C \cong \mathbb{F}_n$, for $n \geq 1$, I'll write its divisor classes as $ae_n + bf_n$, where e_n is the $(-n)$ -section and f_n is a fiber.

Sometimes it is usefull to map curves in E_C birationally to \mathbb{P}^2 . This will be done in the following way:

Lemma 1.1.1. *Let D be a curve in $E \cong \mathbb{F}_n$. Then there is a birational map that maps D to a curve $C \subset \mathbb{P}^2$ such that:*

- (i) *If $n = 0$ and $D \equiv (a, b)$, then C has degree $a + b$, one point of multiplicity a and one point of multiplicity b .*
- (ii) *If $n > 0$ and $D \equiv ae_n + bf_n$, then C has degree b , a point p of multiplicity $b - a$ and $n - 1$ points of multiplicity a infinitely near to p . The points of D lying on e_n are mapped to points infinitely near to p .*

Proof. In item (i), just consider the map that blows up a general point of \mathbb{F}_0 and contracts the two lines through it.

To prove (ii), for $n = 1$ take the map contracting the (-1) -section. For $n > 1$, consider the elementary transformation $\varepsilon : \mathbb{F}_n \dashrightarrow \mathbb{F}_{n-1}$, consisting of the blow up at a general point not contained in e_n and the contraction of the strict transform of the fiber through this point. It maps a curve of class $ae_n + bf_n$ to a curve of class $ae_{n-1} + bf_{n-1}$ and it creates a point of multiplicity a in e_{n-1} . Then the result follows. □

After mapping curves in E_C to curves in \mathbb{P}^2 , we often perform suitable Cremona transformations, in order to obtain simpler curves. Those that will be used the most are the *standard quadratic transformations*:

Definition 1.1.2. Let p_1, p_2, p_3 be three points in \mathbb{P}^2 . The standard quadratic transformation based in p_1, p_2, p_3 is the Cremona transformation in \mathbb{P}^2 determined by the linear system of conics through these points. It blows up p_1, p_2, p_3 and contracts the three lines containing pairs of these points.

A curve of degree d and multiplicity m_i in p_i is mapped via such transformation to a curve of degree $2d - m_1 - m_2 - m_3$ with multiplicity $d - m_j - m_k$ in the contraction of the line through p_j and p_k .

We now recall the definition of tangent cone. Let $Y \subset \mathbb{A}^n$ be an affine variety defined by an ideal I and let y a point in Y . Consider all the polynomials in I expanded around y and let I^* be the ideal generated by the *leading forms* of these polynomials, that is, the homogeneous parts of lower degree. Then the *affine tangent cone* of Y in y is the scheme defined by the ideal I^* , that is, it is the spectrum of the ring:

$$\mathbb{C}[X_1, \dots, X_n]/I^* \cong \bigoplus_{n \geq 0} \frac{m_y^n}{m_y^{n+1}}$$

where X_1, \dots, X_n are the coordinates in \mathbb{A}^n and (\mathcal{O}_y, m_y) is the local ring of regular functions of Y in y .

If $Y \subset \mathbb{P}^n$ is a projective variety, the *projective tangent cone* of Y in y , denoted $\mathcal{C}_y Y$ is the closure of the respective affine tangent cone in an affine chart containing y . We will use the term *tangent cone* for the projective definition, as the varieties studied here are usually projective.

The dimension of $\mathcal{C}_y Y$ is equal to the local dimension of Y in y and its degree is equal to the multiplicity of Y in y . Since it is defined by homogeneous polynomials, it can be projectivized. Its projectivization is isomorphic to the exceptional divisor of the blow up of Y in y .

The following is an easy result on tangent cones:

Lemma 1.1.3. *Let Y, Z be two hypersurfaces in \mathbb{P}^n and let $q \in Y \cap Z$ be a point such that:*

$$\text{mult}_q(Y \cap Z) = (\text{mult}_q Y) \cdot (\text{mult}_q Z)$$

Then:

$$\mathcal{C}_q(Y \cap Z) = \mathcal{C}_q Y \cap \mathcal{C}_q Z$$

Proof. Let \mathbb{A}^n be an affine chart of \mathbb{P}^n containing q . Then Y and Z are defined in \mathbb{A}^n respectively as the zero locus of polynomials f and g in n variables. Let f^* and g^* be the leading forms of these polynomials expanded around q and let I be the ideal generated by f^* and g^* .

Then $\mathcal{C}_q Y$ is defined by the ideal (f^*) , $\mathcal{C}_q Z$ is defined by (g^*) and $\mathcal{C}_q Y \cap \mathcal{C}_q Z$ is defined by I . The tangent cone of $Y \cap Z$ is generated by the leading forms of polynomials in the ideal (f, g) . These include the polynomials in I . Therefore:

$$\mathcal{C}_q(Y \cap Z) \subset (\mathcal{C}_q Y \cap \mathcal{C}_q Z)$$

The hypothesis implies that the $\mathcal{C}_q Y$ and $\mathcal{C}_q Z$ do not have common components. Therefore:

$$\dim(\mathcal{C}_q Y \cap \mathcal{C}_q Z) = n - 2 = \dim \mathcal{C}_q(Y \cap Z)$$

Moreover, the hypothesis also implies that:

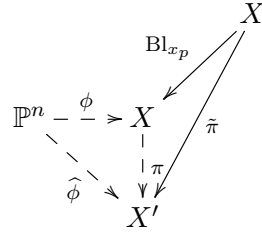
$$\deg \mathcal{C}_q(Y \cap Z) = \deg \mathcal{C}_q Y \cdot \deg \mathcal{C}_q Z = \deg(\mathcal{C}_q Y \cap \mathcal{C}_q Z)$$

Hence the result is proved. □

Let $\phi : \mathbb{P}^n \dashrightarrow X \subset \mathbb{P}^N$ be a birational map defined by a linear system \mathcal{L} in \mathbb{P}^n . Suppose there is a point $p \in \mathbb{P}^n$ of multiplicity $m > 0$ for \mathcal{L} that is mapped to a point $x_p \in X$. In other words, blowing up p , the strict transform of \mathcal{L} intersects $E_p \cong \mathbb{P}^{n-1}$ in a fixed divisor of degree m . The following Lemma gives a method to understand the projectivized tangent cone of X at x_p , that is, the exceptional divisor of the blow up of X in x_p .

Lemma 1.1.4. *Keeping the above notation, let $\widehat{\mathcal{L}}$ be the linear system of hypersurfaces in \mathcal{L} having multiplicity $m+1$ in p . Let C be the image of E_p via the map defined by the moving part of $\widehat{\mathcal{L}} \cap E_p$. If E_p is not contracted by this map, that is, if C has dimension $n-1$, then it is projectively equivalent to a component of the projectivized tangent cone of X at x_p .*

Proof. The hypersurfaces in \mathcal{L} correspond, in \mathbb{P}^N , to the hyperplane sections of X . Since p is mapped to x_p , $\widehat{\mathcal{L}}$ corresponds to those hyperplane sections which contain x_p . Then $\widehat{\mathcal{L}}$ defines a map $\widehat{\phi} : \mathbb{P}^n \dashrightarrow X' \subset \mathbb{P}^{N-1}$, where X' is the image of X via the internal projection π from x_p . Note that $\pi^{-1}(C) = x_p$. Let $\tilde{\pi}$ be the resolution of π by the blow up $\tilde{X} \rightarrow X$ of X in x_p , with exceptional divisor $E \subset \mathbb{P}^{N-1}$, where \mathbb{P}^{N-1} is the exceptional divisor of the blow up of \mathbb{P}^N in x_p . See the diagram below:



The exceptional divisor E is projectively equivalent to its image $C' \subset X'$ via $\tilde{\pi}$, that is, C' is projectively equivalent to the projectivized tangent cone of X in x_p . Note that the cone over C' with vertex x_p is actually the tangent cone of X in x_p , since it consists of the lines having an extra tangency with X in x_p .

Since π^{-1} contracts C to x_p , it follows that $C \subset C'$. By hypothesis, C has dimension $n - 1$, so it is a component of C' and the result follows. \square

To illustrate this Lemma, we'll give an example:

Example 1.1.5. Let \mathcal{L} be the linear system of cubic curves in \mathbb{P}^2 having two base points: p and q , with $p \prec q$. It defines a map $\phi : \mathbb{P}^2 \dashrightarrow X \subset \mathbb{P}^7$. It is known that X has a singularity of type A_1 , which is locally of the form $x^2 + y^2 + z^2 = 0$. This singular point x_p is the image of p .

Let $\widehat{\mathcal{L}}$ be the linear system of curves in \mathcal{L} having multiplicity two in p . Note that p is the only base point of $\widehat{\mathcal{L}}$. It defines the map $\widehat{\phi} : \mathbb{P}^2 \dashrightarrow X' \subset \mathbb{P}^6$, where $X' = \pi(X)$ is the projection of X from x_p . The point x_p is the contraction via the inverse of π of a conic $C \subset X'$: the cone over C with vertex x_p is the tangent cone of X in x_p .

Blowing up the point p , the restriction of $\widehat{\mathcal{L}}$ to E_p has degree two and no base points, so it is mapped by $\widehat{\phi}$ to a conic. By Lemma 1.1.4, this conic is C .

Now let \mathcal{M} be the linear system of cubic curves in \mathbb{P}^2 having three infinitely near base points, that is, $p \prec q \prec q'$. It defines a map $\psi : \mathbb{P}^2 \dashrightarrow Y \subset \mathbb{P}^6$ which maps p to a point y_p .

Let $\widehat{\mathcal{M}}$ be the linear system of curves in \mathcal{M} having multiplicity two in p . These curves have a further base point in q . Therefore, blowing up p , the moving part of $\widehat{\mathcal{M}} \cap E_p$ maps it to a line. By Lemma 1.1.4, the projectivized tangent cone of Y in y_p contains a line, that is, this tangent cone contains a plane.

This agrees with the fact that y_p is a singularity of type A_2 , which is locally of the form $x^2 + y^2 + z^3 = 0$. Its tangent cone is $x^2 + y^2 = 0$, which factors as the union of two planes.

1.2 Bronowski and OADP varieties

Let $X \subset \mathbb{P}^N$ be an irreducible, non degenerate, projective variety of dimension n . Let $X(2)$ be the twofold symmetric product of X . Consider the incidence correspondence:

$$I(X) := \{((x_1, x_2), p) \in X(2) \times \mathbb{P}^N : x_1 \neq x_2, p \in \langle x_1, x_2 \rangle\}$$

Define the *abstract secant variety* of X to be the closure $\mathcal{S}(X)$ of $I(X)$ in $X(2) \times \mathbb{P}^N$. The image $\text{Sec}(X)$ of the projection of $\mathcal{S}(X)$ to \mathbb{P}^N is the *secant variety* of X . In other words, $\text{Sec}(X)$ is the closure in \mathbb{P}^N of the union of the *secant lines* of X , that is, the lines spanned by two distinct points in X .

Since $\mathcal{S}(X)$ has dimension $2n + 1$, the surjective map $p_X : \mathcal{S}(X) \rightarrow \text{Sec}(X)$ gives:

$$\dim(\text{Sec}(X)) \leq \min\{2n + 1, N\}$$

and X is said to be *secant defective* if the strict inequality holds. According to *Terracini's Lemma*, X is secant defective if and only if:

$$\dim(T_{x_1}X \cap T_{x_2}X) > \min\{2n - N, -1\}$$

A *general tangential projection* of $X \subset \mathbb{P}^{2n+1}$ is a projection $\tau_x : X \dashrightarrow \mathbb{P}^n$ from the tangent space T_xX at a general point $x \in X$. Then X is said to be a *Bronowski variety* if its general tangential projection is birational.

A Bronowski variety is not secant defective. Indeed, let C be the join of T_xX and X . Then by [Ru2, equation 1.2.8]:

$$\dim(C) = \dim(T_xX) + \dim(X) - \dim(T_xX \cap T_yX)$$

with y a general point in X . If X is Bronowski, τ_x is dominant and $\dim(C) = 2n + 1$, giving $\dim(T_xX \cap T_yX) = -1$. Then the result follows by Terracini's Lemma.

Suppose now that $X \subset \mathbb{P}^{2n+1}$ is not secant defective, that is, $\text{Sec}(X) = \mathbb{P}^{2n+1}$. Then we say that X is a *variety with one apparent double point*, shortly *OADP variety*, if $p_X : \mathcal{S}(X) \rightarrow \mathbb{P}^{2n+1}$ is birational, that is, if through a general point of \mathbb{P}^{2n+1} there is a unique secant line to X .

Using the same argument of [CMR, §3], we have that *OADP varieties are Bronowski varieties*. The following is known as the *Bronowski conjecture*:

Conjecture 1.2.1. *Any Bronowski variety is OADP.*

1.3 The tangential projection

Let $X \subset \mathbb{P}^7$ be a Bronowski threefold. The results in this section (except for Lemma 1.3.3) were presented in [CR] for arbitrary $n = \dim X$.

Let x be a general point of X and let $\tau = \tau_x : X \dashrightarrow \mathbb{P}^3$ be the associated tangential projection. Consider the blow up $\pi : \text{Bl}_x(X) \rightarrow X$ of X at x with exceptional divisor $E \cong \mathbb{P}^2$ and set $\tilde{\tau} = \tau \circ \pi$. Let V be the image of E via $\tilde{\tau}$. Then the map:

$$\bar{\tau} = \tilde{\tau}|_E : E \dashrightarrow V \subset \mathbb{P}^3$$

is birational (see [GH, (5,7)]). It is defined by a linear system of conics, the *second fundamental form* $\Pi_{X,x}$ of X at x . See [IL, (3.2)] for other definitions of the second fundamental form.

The surface V is called the *fundamental surface* of X . Since it is the image of \mathbb{P}^2 via a linear system of conics, V is a birational projection to \mathbb{P}^3 of the Veronese surface of degree four in \mathbb{P}^5 . Hence:

Lemma 1.3.1. *If X is a Bronowski threefold, then one of the following holds:*

- (i) $\Pi_{X,x}$ has a fixed line and V is a plane;
- (ii) $\Pi_{X,x}$ has three base points and V is a plane;
- (iii) $\Pi_{X,x}$ has two base points and V is a smooth quadric surface or a quadric cone;
- (iv) $\Pi_{X,x}$ has one base point and V is a cubic surface, projection of a cubic scroll from an external point;
- (v) $\Pi_{X,x}$ has no base points and V is a Steiner quartic surface, projection of the Veronese quartic from an external point.

Cases (i) and (ii) have been completely described in [CMR]. Case (iii) will be considered in Chapter 2, case (iv) in Chapter 3 and case (v) in Chapter 4.

Let σ be the inverse of τ , defined by a linear system \mathcal{X} , of dimension 7 and degree $d \leq \deg(X)$. It contracts V to the point $x \in X$, so \mathcal{X} has fixed intersection with V .

Set $\delta = \deg(V)$. Let \mathcal{X}' be the linear system of surfaces F of degree $d - \delta$ such that $F + V \in \mathcal{X}$. The moving part of \mathcal{X}' corresponds, via τ , to hyperplane sections of X through x .

And let \mathcal{X}'' be the linear system of surfaces F of degree $d - 2\delta$ such that $F + 2V \in \mathcal{X}$. Its moving part corresponds, via τ , to tangent hyperplane sections of X at x , that is, hyperplane sections of X containing $T_x X$. In conclusion:

Lemma 1.3.2. *Let \mathcal{X}' and \mathcal{X}'' be as defined above. Then:*

- (i) *The image of \mathbb{P}^3 via the map defined by \mathcal{X}' is the image of X by the internal projection $\pi_x : X \rightarrow X' \subset \mathbb{P}^6$ from x .*
- (ii) *The restriction of \mathcal{X}' to V defines the map $\bar{\sigma}$, inverse of $\bar{\tau}$*
- (iii) *The moving part of \mathcal{X}'' is the linear system of planes in \mathbb{P}^3 , which correspond, via τ , to the tangent hyperplane sections of X at x .*
- (iv) *If $\Pi \subset \mathbb{P}^3$ is a plane containing a base curve C of \mathcal{X} , then it corresponds to a tangent hyperplane section of X containing the image of the blow up at C .*
- (v) *The fixed part of \mathcal{X}'' consists of the exceptional divisors corresponding to the blow up at the indeterminacy locus of τ out of x and not contracted by τ .*

Item (iii) implies that $d \geq 2\delta + 1$ and equality holds if and only if \mathcal{X}'' has no fixed part. In this thesis, the following hypothesis will be made on X :

- (H) Let X be a normal Bronowski threefold such that the linear system \mathcal{X} defining the inverse of the tangential projection at a general point $x \in X$ has degree $d = 2\delta + 1$, where δ is the degree of the fundamental surface.

This is a working hypothesis, so we are restricting to a class of normal Bronowski threefolds satisfying this condition on the degree of \mathcal{X} . We now make some remarks in order to clarify its meaning.

As noted in Lemma 1.3.2, $d > 2\delta + 1$ if and only if \mathcal{X}'' has a fixed part, which consists of the exceptional divisors corresponding to the blow up at the indeterminacy locus of τ out of x . So, one has to analyse the intersection scheme $T_x X \cap X$ and decide if its components, with the exception of x , are contracted to curves or points in \mathbb{P}^n .

Since $\dim X = 3$, by [CMR, Proposition 5.2] $T_x X \cap X$ has a component of dimension two if and only if X is either a hypersurface or a scroll over a curve. Moreover, it was explained in [CR, Proposition 5.5] that a normal Bronowski

threefold that is a scroll over a curve is a smooth rational normal scroll. In this case, \mathcal{X} is the linear system of cubic surfaces having a double and a simple fixed lines, both lying in the plane V . Therefore, if X is a normal Bronowski threefold, hypothesis (H) does not hold only if $\dim(T_x X \cap X) \leq 1$.

Note also that (H) is satisfied when X is a smooth OADP threefold. This can be proven by analysing the tangential projection of the threefolds given in the classification in [CMR]. Another option is to use the fact that in this case the support of $T_x X \cap X$ is either x or a bunch of lines through x (see [CMR], Lemma 5.3 and Proposition 5.4). Moreover, by [CMR, Proposition 6.5], a line of X is mapped by τ to a line in \mathbb{P}^3 . Therefore, the image of the blow up of the indeterminacy locus of τ off x is one-dimensional, so $d = 2\delta + 1$.

If these considerations can be generalized for a smooth Bronowski threefold, then the results in this thesis imply that the Bronowski conjecture holds for smooth threefolds.

The condition $d = 2\delta + 1$ is also satisfied when X is a normal Bronowski variety in which the fundamental surface is a plane, that is, $\delta = 1$. This follows from the remarks made above on scrolls over a curve and from [CR, Proposition 2.3 and Proposition 5.7].

Before proceeding to the next result, we'll say some words on the dual variety. If $X \subset \mathbb{P}^r$ is an irreducible variety, the *dual variety* of X , denoted X^* is the image in \mathbb{P}^{r*} of the *conormal scheme*, which is the Zariski closure in $X \times \mathbb{P}^{r*}$ of the irreducible scheme:

$$\{(x, H) : x \in X \setminus \text{Sing}(X), H \in \mathbb{P}^{r*}, T_x X \subset H\}$$

of dimension $r - 1$. Therefore $\dim(X^*) \leq r - 1$, and X is said to be *dual defective* if strict inequality holds. According to [GH, (3.5)], X is dual defective if and only if at a general point $x \in X$, the second fundamental form $\Pi_{X,x}$ is a singular linear system, that is, every member is singular.

These definitions can be extended, component-wise, for reducible varieties.

The following Lemma will be useful in understanding the singularities of X .

Lemma 1.3.3. *Let X be a Bronowski threefold, $x \in X$ a general point and $\sigma : \mathbb{P}^3 \dashrightarrow X$ be the inverse of τ , as defined above. Suppose that $\Pi_{X,x}$ has general smooth member. Let $q \in \mathbb{P}^3$ be a point that is the preimage via σ of a point $x_q \in X$. If the image of a general plane through q has a singularity of multiplicity m in x_q , then X has a singularity of multiplicity m in x_q .*

Proof. By Lemma 1.3.2, planes in \mathbb{P}^3 are mapped by σ to tangent hyperplane sections of X at x . Since x_q is projected to q , tangent hyperplane sections of X at x containing x_q are projected to planes in \mathbb{P}^3 through q .

Since the second fundamental is not singular at a general point of X , it follows that $\dim(X^*) = 6$. The tangent hyperplane sections of X (at any point) that contain x_q correspond to points in a hyperplane section $\Sigma \cap X^* = \Sigma_X$ of X^* , with $\Sigma \subset (\mathbb{P}^7)^*$.

By hypothesis, the image of a general plane through q has a singularity of multiplicity m in x_q . Then this is a point of X of multiplicity $m' \leq m$. Suppose that equality does not hold, that is $m' < m$. Then a general tangent hyperplane section of X at x through x_q is tangent to the tangent cone C of X in x_q . Since x is general, it can be replaced by any general point, that is, this holds for a hyperplane section corresponding to a general point in Σ_X .

The threefold C is a cone over a (possibly reducible) surface Y with vertex x_q . Therefore a hyperplane is tangent to C in x_q if and only if it contains x_q and is tangent to Y . Such hyperplanes correspond to points in the hyperplane section $\Sigma \cap Y^* = \Sigma_Y$ of Y^* . Note that $\dim(\Sigma_Y) \leq 5$.

Since a general tangent hyperplane section of X through x_q is also tangent to C , it follows that $\Sigma_X \subset \Sigma_Y$. Therefore Σ_X is a component of Σ_Y , that is, there is a component Z of Y such that $\Sigma_X = \Sigma_Z := \Sigma \cap Z^*$.

Hence, tangent hyperplane sections of X through x_q coincide with hyperplane sections of X through x_q that are tangent to $Z \subset Y$. Remember that the cone over Y with vertex x_q is the tangent cone of X at this point. Then, if X' is the projection of X from x_q to a \mathbb{P}^6 containing Y , it follows that $Y \subsetneq X'$, since X' is a threefold (X is not a cone). Moreover, tangent hyperplane sections of X' coincide with hyperplane sections of X' that are tangent to the surface $Z \subset X'$. This implies $(X')^* = Z^*$.

Dualizing this equality gives $X' = Z$ (see [Ha], Theorem 15.24 and Example 16.20), a contradiction. Thus $m = m'$.

□

1.4 OADP threefolds

I will now quickly recall two examples of OADP threefolds that will be used in the following chapters. A more detailed description of these and other examples, as well as the proofs that these are OADP threefolds, can be found in [CMR].

Example 1.4.1 (Degree seven Edge threefolds). Let $Y \subset \mathbb{P}^7$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$. Let Π be a general \mathbb{P}^3 of the ruling of Y and Q a

quadric of \mathbb{P}^7 containing Π . Then the residual intersection of Q with Y is a smooth threefold, called the *Edge threefold of degree seven*. It is part of a series of OADP varieties with dimension $n \geq 2$ and degree $2n + 1$ defined by Edge, in [Ed], and Babbage, in [Ba]. In the same paper, Edge also defines another series of OADP varieties, which have degree $2n$.

The degree seven Edge threefold has two lines through a general point. In Chapter 2, it will be showed that the fundamental surface of X is a smooth quadric.

Example 1.4.2 (Degree eight scroll in lines). This example was first proven to be an OADP variety by Alzati and Russo, in [AR1]. See also [CMR]. Let Y be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ and let $F \subset \mathbb{P}^7$ be the cone over Y having a line as vertex. Define X to be the residual intersection of F with two general quadrics containing a \mathbb{P}^4 of its ruling.

This threefold has degree eight and is ruled by lines. In Chapter 3, we'll see that $\Pi_{X,x}$ has one base point and the fundamental surface V is a cubic.

Note that these two OADP threefolds are defined by the residual intersection of quadrics with a rational normal scroll. Related to this, is the following definition: An irreducible non-degenerate variety $X \subset \mathbb{P}^N$ is *linearly normal* if one of the following equivalent properties holds:

- (i) there is no variety $X' \subset \mathbb{P}^{N'}$ with $N' > N$ and a linear projection $\phi : \mathbb{P}^{N'} \dashrightarrow \mathbb{P}^N$ inducing an isomorphism $\phi : X' \rightarrow X$;
- (ii) the linear system $|\mathcal{O}_X(1)|$ has dimension N .

Remark 1.4.3. It was proven in [CR] that OADP varieties are linearly normal. In the case of a Bronowski variety $X \subset \mathbb{P}^{2n+1}$, let \mathcal{X} be the linear system defining the inverse of a general tangent projection. Suppose \mathcal{X} is *relatively complete*, that is, it is the complete linear system on \mathbb{P}^n with the same class in $\text{Pic}(\mathbb{P}^n)$ and with the same base locus of \mathcal{X} . Then the strict transform of \mathcal{X} via τ , which is the complete linear system $|\mathcal{O}_X(1)|$, has dimension $2n + 1$. Then X is linearly normal.

In order to understand the varieties constructed here as subvarieties of a rational normal scroll, the following theorem (see [EH]) will be used:

Theorem 1.4.4. *Let $X \subset \mathbb{P}^N$ be a linearly normal variety, and $D \subset X$ a divisor. If D moves in a pencil $\{D_\lambda | \lambda \in \mathbb{P}^1\}$ of linearly equivalent divisors, then writing \overline{D}_λ for the linear span of D_λ in \mathbb{P}^N , the variety:*

$$S = \bigcup_{\lambda} \overline{D}_\lambda$$

is a rational normal scroll.

Chapter 2

The quadric case

This chapter treats the case in which the fundamental surface V is a quadric. This is the situation of the Edge threefold of degree seven, described in Example 1.4.1.

2.1 First Considerations

2.1.1 The base locus of \mathcal{X}

The main result of this section is the following:

Lemma 2.1.1. *The linear system \mathcal{X} is defined as degree five surfaces having:*

- *multiplicity two in C_4 , a quadric section of V ;*
- *multiplicity two at a smooth point p of V ;*
- *multiplicity one in ℓ and ℓ' , the two (possibly infinitely near) lines in V through p .*

In particular:

$$\mathcal{X} \cap V = 2C_4 + \ell + \ell'$$

If V is a cone with vertex q , then \mathcal{X} has multiplicity three or four at q . If it is three, there is a quadric containing C_4 which is smooth in q . If it is four, all such quadrics are singular in this point.

Proof. Since $\deg V = 2$, the map $\bar{\tau} : E \dashrightarrow V$ is defined by conics with two fixed points. Its inverse is defined by plane sections of V through a fixed smooth point p . The quadric V is a cone if and only if the two base points of $\bar{\tau}$ are infinitely near.

By hypothesis (H), $\deg(\mathcal{X}) = 5$. Then the restriction of \mathcal{X}' to V gives cubic sections of V . The moving part is made of plane sections through p . So the fixed part is a quadric section C_4 of V .

Being a fixed curve of \mathcal{X}' , C_4 is a double curve of \mathcal{X} . So the restriction of \mathcal{X} to V is C_4 with multiplicity two and a fixed plane section of V .

Since p is in the base locus of \mathcal{X}' , it is a double point of \mathcal{X} . Note that, removing V twice from \mathcal{X} gives the linear systems of planes in \mathbb{P}^3 , so \mathcal{X} has not multiplicity greater than two in C_4 or in p .

If $p \notin C_4$, the degree two curve in $\mathcal{X} \cap V$ is a pair of (possibly coincident) lines through p .

If $p \in C_4$, then $\mathcal{X}' \cap V$ has at least a double point at p . But, as noted above, p cannot be a double point of \mathcal{X}' . So \mathcal{X}' is tangent to V at p and, hence, \mathcal{X} has multiplicity two in $p \in C_4$ and in an infinitely near point to p . Then again the degree two curve is a pair of lines through p .

This proves the first part. Note that the case where $p \in C_4$, the point infinitely near to p cannot lie on C_4 . Indeed, this would imply that \mathcal{X}' has multiplicity three in p , which would then be a base point of \mathcal{X}'' . This contradicts hypothesis (H).

Suppose now that V is a cone and let q be its vertex. So $\ell = \ell' = \langle p, q \rangle$ (remember that p is a smooth point of V). Since \mathcal{X}' must desingularize V , $\text{mult}_q \mathcal{X}' \geq 1$. Reattaching V we find that:

$$\text{mult}_q \mathcal{X} \geq 3$$

On the other hand, it can't be greater than four, since \mathcal{X}'' has no base points.

If $\text{mult}_q \mathcal{X} = 4$, then $\text{mult}_q \mathcal{X}' = 2$, so C_4 has multiplicity four in q and breaks into four lines. In particular, a quadric intersecting V in C_4 is another cone with vertex q .

If $\text{mult}_q \mathcal{X} = 3$, then q is a double point of C_4 . In this case, a quadric intersecting V in C_4 is not a cone with vertex q , otherwise C_4 would be the union of four lines through q and \mathcal{X} would have multiplicity four at q . Then there is a quadric containing C_4 which is smooth in q . □

If V is a smooth quadric, ℓ and ℓ' are obviously the pair of lines through p in V . So, in this case, different Bronowski threefolds are determined by the choice of C_4 and p (they define a unique quadric V). This is also the case if V is a cone, with the difference that the lines through p are infinitely near.

Since C_4 is a quadric section of V , it is the base locus of a pencil of quadrics \mathcal{Q} .

Lemma 2.1.2. *Each quadric of \mathcal{Q} is mapped isomorphically by σ to a quadric contained in X . The point p is mapped to a quadric S_2^x in X through x , isomorphic to V . These form a family \mathcal{Q}' of disjoint quadric surfaces that cover X .*

Proof. The restriction of \mathcal{X} to a quadric of \mathcal{Q} (except for V) is the double curve C_4 and moving plane sections. So it is mapped in \mathbb{P}^7 to a isomorphic quadric contained in X . Two quadrics of \mathcal{Q} are disjoint after the blow up at C_4 .

Blowing up p , \mathcal{X}_p is a linear system of conics with two base points, given by the directions of ℓ and ℓ' at p . These points are infinitely near if and only if V is a cone. So p is mapped to a quadric isomorphic to V . Since V_p is the line through the two base points of \mathcal{X}_p , this quadric contains x . The only quadric of \mathcal{Q} through p is V , so the quadrics in \mathcal{Q}' are disjoint.

Given a point $y \in X$, it is the image of a point in $y' \in \mathbb{P}^3$ or it lies on the exceptional locus of σ . In the first case, there is a quadric of \mathcal{Q} through y' which is mapped to a quadric of \mathcal{Q}' through y . If y is the image of $y' \in E_{C_4}$, again there is a quadric of \mathcal{Q} through this point (if this quadric is V , then $y = x$). The point p is mapped to S_2^x and, as it will be noted in Lemma 2.2.2, ℓ and ℓ' are mapped to lines in S_2^x . This finishes the proof. \square

Remember that planes in \mathbb{P}^3 correspond to tangent hyperplane sections of X at x . \mathcal{X} intersects a general plane in curves of degree five having four double and two simple points, mapping it to a surface of degree:

$$25 - 4 \cdot 4 - 2 = 7$$

As a consequence, X has degree seven.

2.1.2 Equations for \mathcal{X}

We now give general equations for \mathcal{X} . Suppose V is given by $(g = 0)$ and take another quadric of \mathcal{Q} with equation $(g' = 0)$. An equation for \mathcal{X} is then:

$$(g)^2(a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3) + gg' \cdot P_p + a_7g'h = 0$$

where $P_p = 0$ is a general equation for the linear system of planes through p (so it has three parameters) and $(h = 0)$ is a cubic surface not containing

V that has a double point at p and contains C_4 . Such a cubic can be taken from the linear system:

$$g(b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3) + g'(c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3) = 0$$

by imposing the condition of singularity in p .

Note that (after removing the common factor) the first four monomials give \mathcal{X}'' , and the first seven give \mathcal{X}' .

Setting $p = (1 : 0 : 0 : 0)$, the map σ can be written as:

$$\sigma = (g^2x_0 : g^2x_1 : g^2x_2 : g^2x_3 : gg'x_1 : gg'x_2 : gg'x_3 : g'h)$$

2.1.3 A Cremona transformation

We now start with a curve C_4 and a point $p \notin C_4$ (possibly infinitely near to a point of C_4) in \mathbb{P}^3 , and suppose that C_4 is the base locus of a pencil of quadrics \mathcal{Q} with general smooth member. Suppose also that the quadric $V \in \mathcal{Q}$ containing p is irreducible and smooth in p . If V is smooth, let ℓ and ℓ' be the lines in V through p . If V is a cone let $\ell = \ell'$ be the line joining p and the vertex of V .

Let \mathcal{L} be the linear system of cubic surfaces containing C_4 and having a double point at p . If p is infinitely near to $\hat{p} \in C_4$, define \mathcal{L} containing C_4 , having a double point at \hat{p} and with tangent cone at \hat{p} containing the tangent plane of V in \hat{p} .

In both cases, those cubic surfaces contain the lines ℓ and ℓ' and intersect V at $C_4 + \ell + \ell'$.

Lemma 2.1.3. *Suppose that the conditions imposed on \mathcal{L} are independent. Then \mathcal{L} defines a Cremona transformation:*

$$f_{\mathcal{L}} = f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$$

It maps p to a quadric V' isomorphic to V and contracts V to a point $p' \in V$. It maps the other quadrics of \mathcal{Q} to quadrics of a pencil having a base curve C'_4 of the same type of C_4 .

Proof. Cubic surfaces containing C_4 have equations $(gf_1 + g'f_2 = 0)$, where $(g = 0)$ and $(g' = 0)$ are equations for two quadrics of \mathcal{Q} and f_1, f_2 are linear forms; so there are eight parameters. If $p \notin C_4$, the number of conditions for surfaces to have a double point in p is four. If p is infinitely near to $\hat{p} \in C_4$, two conditions are needed for a double point in \hat{p} and two more conditions for the tangent cone of \mathcal{L} in \hat{p} to contain $T_{\hat{p}}V$. Since the conditions are

independent, this linear system has projective dimension three. So f maps to \mathbb{P}^3 .

The union of V and planes through p give three independent cubics in \mathcal{L} . Hence \mathcal{L} is generated by these three cubics plus another cubic not containing V . So the intersection of two surfaces of \mathcal{L} is made of C_4, ℓ, ℓ' and plane sections of this cubic through p . And two plane sections through p intersect at a single point besides p itself. Hence f is a Cremona transformation.

The contraction of V is obvious, since \mathcal{L} has fixed intersection with it. Blowing up p , \mathcal{L} cuts E_p in conics with two base points (just like what happened with \mathcal{X}). So f maps p to a quadric surface V' , which is a cone if and only if V is.

A quadric of \mathcal{Q} is cut by \mathcal{L} in the fixed curve C_4 and a linear system of plane sections. So it is mapped again to an isomorphic quadric surface. Hence, the image of \mathcal{Q} is a pencil having a base curve C'_4 of the same type of C_4 . □

The map f is a Cremona transformation of De Jonquieres type: it restricts to planes through p in the linear system of cubics with one double point and four simple points. These planes are mapped to planes through p' .

The intersection of \mathcal{L} with a general plane consists of cubic curves with six simple points. Therefore this plane is mapped by f to a cubic surface, that is, f^{-1} is also defined by cubic surfaces.

The linear system defining f^{-1} has fixed intersection with V' and a double point in $p' = f(V)$. This fixed intersection is C'_4 and the two lines in V' through p . Indeed, f^{-1} maps each quadric of the pencil having base locus C'_4 to a quadric in \mathcal{Q} , so this curve lies on the base locus of f^{-1} . Therefore:

Lemma 2.1.4. *The rational map f^{-1} is defined by the linear system of cubic surfaces containing C'_4 and having a double point in p' , the image of V via f .*

2.1.4 The Segre symbol of a pencil of quadrics

Since each quadric of the pencil \mathcal{Q} is mapped isomorphically to a quadric of \mathcal{Q}' , a good tool to understand X is the *Segre symbol* of \mathcal{Q} . It is due to a classification of C. Segre of pencils of quadrics, done in [Se2]. The following quick revision is based on a paper of D. Avritzer and R. Miranda [AM].

Suppose the pencil \mathcal{Q} is given by:

$$\lambda F_1 + \mu F_2 = 0$$

Let A_1 and A_2 be the two 4×4 symmetric matrices defining the quadratic forms F_1 and F_2 and consider the equation:

$$\det(\lambda A_1 + \mu A_2) = 0 \quad (2.1)$$

Suppose there is a smooth quadric in \mathcal{Q} . This implies that the left-hand side of (2.1) is not identically zero, so it has 4 roots in $(\lambda : \mu)$, counting multiplicities. These roots correspond to the singular quadrics in \mathcal{Q} , being in general 4 cones.

In some cases, a root of (2.1) is also a solution of the equations on the 3×3 minors, producing a pair of planes. If it's also a solution of the equations on all 2×2 minors, a double plane is produced.

Given a root of (2.1), let k be the dimension of the singular locus of the quadric determined by this root (so it's a root for the minors of order $4 - k$) and let l_i be the minimum multiplicity with which a root appears in the equations of subdeterminants of order $4 - i$, for $i = 0, \dots, k$. Define $e_i := l_i - l_{i+1} \geq 0$ (set $l_{k+1} = 0$ in order to define e_k). Note that $\sum e_i = l_0$ is the multiplicity of the given root in (2.1).

This gives, for each root $(\lambda_j : \mu_j)$ a sequence $(e_0^j \cdots e_{k_j}^j)$, with $k_j \leq 2$. Let r be the number of distinct roots of (2.1).

Definition 2.1.5. The Segre symbol of the pencil \mathcal{Q} is:

$$[(e_0^1 \cdots e_{k_1}^1), \dots, (e_0^r \cdots e_{k_r}^r)]$$

When $k_j = 0$, the parenthesis are omitted.

For example, a pencil with Segre symbol $[1, 1, 1, 1]$ has four cones, which is the maximum in this case. While the singular members of a pencil having Segre symbol $[(11), 2]$ is a pair of planes and a cone.

Since the quadrics, contained in $X \subset \mathbb{P}^7$, of the family \mathcal{Q}' are isomorphic to the ones of \mathcal{Q} , the Segre symbol of \mathcal{Q} will give us information on the singular members of \mathcal{Q}' . It will help us to describe the singularities of X and to study the different possibilities for the base locus C_4 of \mathcal{Q} . The computations on the Segre symbol and singular elements of \mathcal{Q} done in this thesis can be performed by hand. Alternatively, it can be found in [HP, p. 305].

In [Se2], Segre studies the properties of the base curve of a pencil of quadrics considering its Segre symbol. The following result is a simplified version of this investigation made by Segre. It can also be obtained by a direct analysis of the possible Segre symbols in \mathbb{P}^3 , also done in [HP].

Proposition 2.1.6. *Let q be a singular point of the base locus of a pencil \mathcal{Q} of quadric surfaces in \mathbb{P}^3 with general smooth member. Then q is a singular point of a quadric in \mathcal{Q} corresponding to a root of (2.1) with multiplicity greater than 1.*

2.2 The smooth case

Suppose that the fundamental surface V is a smooth quadric. As noted in Section 2.1.1, the threefold X is determined by the choices of C_4 and p .

Lemma 2.2.1. *Let f be the Cremona transformation of Section 2.1.3 associated to a choice of C_4 and p , and let C'_4 be as in Lemma 2.1.3. The image of \mathcal{X} via f is the linear system of cubic surfaces containing C'_4 . In the general case (C'_4 smooth), this is the rational representation of X described by Edge, in [Ed], and Babbage, in [Ba].*

In particular, the threefold X depends only on the choice of C_4 .

Proof. To find the image of \mathcal{X} via f , first look at \mathcal{X}'' , the planes in \mathbb{P}^3 . Their image is the linear system defining the inverse f^{-1} , that is, cubic surfaces containing C'_4 and having a double point at p' . Since V is contracted to p' and $\mathcal{X}'' = \{S ; S + 2V \in \mathcal{X}\}$, it follows that \mathcal{X} is mapped to cubic surfaces containing C'_4 . □

This Lemma allows us to concentrate our study to the simpler linear system \mathcal{Y} of cubic surfaces containing C_4 , since Lemma 2.1.3 asserts that C'_4 is of the same type of C_4 . We will choose to not follow this path here, since we are interested in the description of the inverse of the general tangential projection of X .

However it is worth to mention the meaning of the map defined by \mathcal{Y} . As it will be proved in Theorem 2.4.1, the threefold X is contained in the Segre embedding Y of $\mathbb{P}^1 \times \mathbb{P}^3$, being the residual intersection of Y with a quadric containing a \mathbb{P}^3 of its ruling. As noted by Edge, the map defined by \mathcal{Y} is the inverse of the birational projection of X from a \mathbb{P}^3 of the ruling of Y to another \mathbb{P}^3 of the ruling. This projection contracts a degree eight scroll, described by the directrix lines of Y that are contained in X , to the curve C_4 (see [Ed] and [Ba]). On the other hand, it will be shown later that the image of the exceptional divisor of C_4 via σ is a degree twelve scroll in conics.

In what follows, we will first study properties of the map σ which are common to all different choices of C_4 . This includes the image of p and the study of conics in X .

Next, we will study how the possible irreducible components of C_4 are blown up and mapped by σ .

After that, the singular locus of X will be analysed. As it will be seen, it arises from the singularities of C_4 . Finally, in Section 2.2.4, the results for which V is smooth will be collected.

2.2.1 Some properties

As noted before, we can suppose that p is a general point in V . We can also suppose that ℓ and ℓ' cut C_4 transversally. If C_4 has only reduced components, ℓ and ℓ' intersect C_4 in two distinct points each.

Lemma 2.2.2. *The point p is blown up and mapped to a smooth quadric surface S_2^x through x and the lines ℓ and ℓ' are mapped respectively to ℓ_x and ℓ'_x , the lines in S_2^x through x .*

Proof. The image of p is explained in Lemma 2.1.2.

Blowing up ℓ gives an exceptional divisor $E_\ell \cong \mathbb{F}_0$. A plane through ℓ cuts \mathcal{X} in ℓ itself plus quartic curves. These cut ℓ in one moving and three fixed points: two in C_4 (name them q and q') and one in p . So:

$$\mathcal{X}_p = M + F_q + F_{q'} + F_p \equiv (4, 1)$$

where $M \equiv (1, 1)$ is the moving part of \mathcal{X}_p and $F_q, F_{q'}, F_p$ are fixed lines, the fibers over these points. The moving part has two base points, lying on F_q and $F_{q'}$, associated to the tangent directions of C_4 at q and q' . Using Lemma 1.1.1, M corresponds, in \mathbb{P}^2 , to conics through four points, so E_ℓ is mapped back to a line. The fixed parts have no moving intersection with \mathcal{X}_p , except for F_p , so they are contracted.

Blowing up p , and then ℓ and ℓ' , we see that these lines are mapped to the lines in S_2^x through x .

If C_4 has a non reduced component, it may happen that the two base points in E_ℓ become infinitely near. However, the result will still be the same.

□

Also note that a general plane through p is cut by \mathcal{X} in quintics with five double points. The conic through these points is the intersection of V with this plane. So it is mapped to a quintic (possibly weak) Del Pezzo

surface through x (this map is the inverse of the tangential projection from x). Remember that planes correspond to tangent hyperplane sections of X in x . Planes through p correspond to those reducible sections containing S_2^x .

Now I make a small remark about conics in X . Since through a general point $x' \in X$ there is a quadric surface contained in X , there is a two-dimensional family of conics through x' : the hyperplane sections of this quadric. The image via τ of such a conic is a conic in the quadric of \mathcal{Q} through $\tau(x')$, cutting C_4 in four points.

There is also an one-dimensional family of conics through x' parametrized by C_4 . Fixed a point in C_4 , consider the plane containing this point, p and $\tau(x')$. It cuts C_4 in three other points. Then there is a conic in this plane passing through these three points, $\tau(x')$ and p . It is mapped to a conic through x' , since \mathcal{X} has multiplicity two in p and along C_4 .

The conics through x of the first family are mapped to the point p , since it is the image of S_2^x . The conics through x of the second family are mapped to the points of C_4 .

Obviously, any of these conics can be a pair of lines or a double line.

2.2.2 The image of irreducible components of C_4

We now study how the irreducible components of C_4 are blown up and mapped by σ . We'll suppose we are not in the general case (where C_4 is an elliptic quartic and $X = E_{3,7}$), so either C_4 is an irreducible rational quartic or each component is a smooth rational curve. Then, after the desingularization, the exceptional divisor of the blow up at C_4 is a union of Hirzebruch surfaces.

Since C_4 is a $(2, 2)$ curve in V , the possible irreducible components are: a rational quartic curve (nodal or cuspidal), a twisted cubic, a conic, a line, a double conic and a double line. Note that the singularities (and double components) of C_4 mean tangency conditions, and not singularities for surfaces in \mathcal{X} .

A rational quartic

If C_4 is a rational quartic, it has a double point q . It can be a nodal or a cuspidal point of C_4 . If it is a nodal point, \mathcal{Q} has Segre symbol $[2, 1, 1]$, if it is cuspidal, the symbol is $[3, 1]$. This can be proven with direct computations. In both cases, the special cone (corresponding to the numbers 2 and 3) of \mathcal{Q} has vertex q .

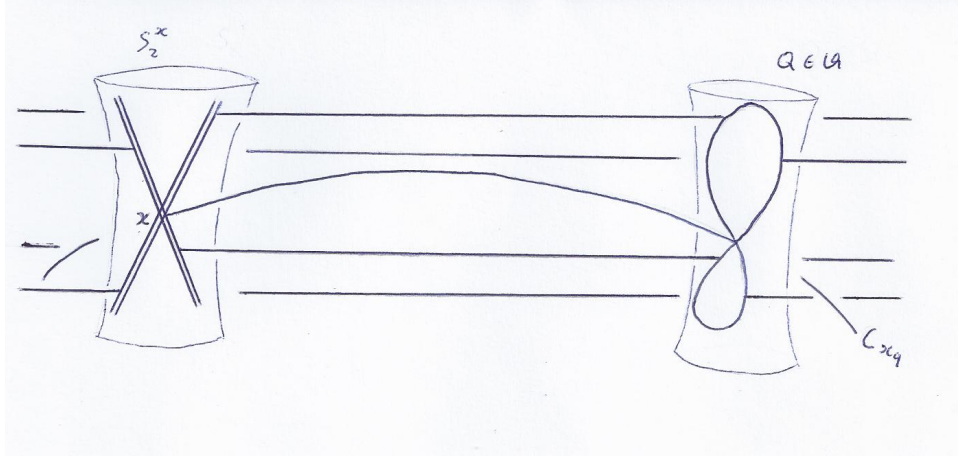


Figure 2.1: The surface ${}^nS_{12}^x$

Proposition 2.2.3. *Suppose C_4 is a rational quartic curve. Then it is mapped to a scroll in conics S_{12}^x of degree 12 having multiplicity four in x , a double conic C_{x_q} and two double lines. It cuts the quadric S_2^x in the two double lines through x and cuts the other quadrics of \mathcal{Q}' in rational quartic curves with a double point. It is a cusp if and only if C_4 is cuspidal in q . These quartics are met by four lines in S_{12}^x , each cutting one of the two double lines through x . The conic C_{x_q} is described by the singular points of these quartic curves.*

To distinguish both cases, denote by ${}^nS_{12}^x$ the image of a nodal quartic and by ${}^cS_{12}^x$ the image of a cuspidal quartic. A sketch of ${}^nS_{12}^x$ is presented in Figure 2.1.

Proof. The blow ups are represented in Figure 2.2.

Before blowing up the curve itself, one must blow up q . \mathcal{X}_q has two double points in $E_q \cong \mathbb{P}^2$, corresponding to the tangent cone of C_4 in q . These points are infinitely near if q is a cusp. So \mathcal{X}_q is a fixed double line connecting these two points, name it t .

The line t has to be blown up too. It is the complete intersection of E_q and V . In E_q , $t^2 = 1$; in V , $t^2 = -1$, since it comes from the blow up of V in q . Hence, the normal bundle of t is:

$$N_t = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

So blowing up t gives $E_t \cong \mathbb{F}_2$, where $V_t \equiv e_2 + 2f_2$ and $(E_q)_t \equiv e_2$. The linear system \mathcal{X}_t must cut each fiber f_2 in two points, E_q in none and V in

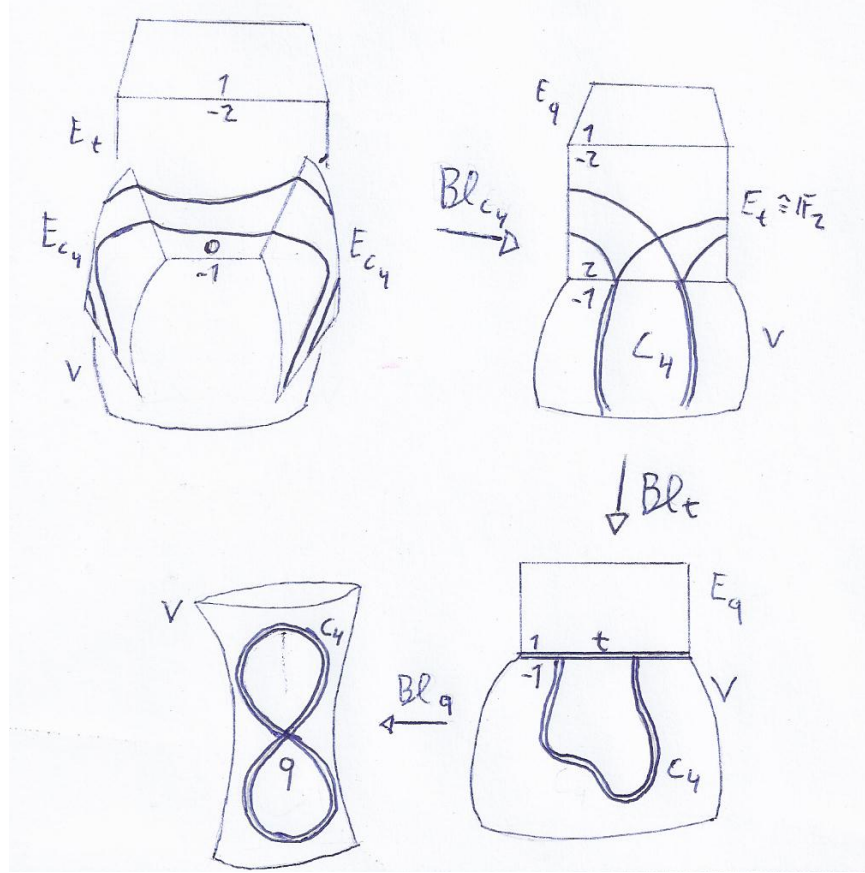


Figure 2.2: \mathcal{X} and the blow up at C_4 in the nodal case

two fixed double points. So:

$$\mathcal{X}_t \equiv 2e_2 + 4f_2$$

with two double base points. As showed in Lemma 1.1.1, it birationally corresponds in \mathbb{P}^2 to degree four curves having two double points, a third double point and a fourth double point infinitely near to this one. A quadratic standard transformation maps it to a system of conics with one double point, that is, pairs of lines through a point. Thus, E_t is sent to a conic in X .

Note that, in the original linear system on \mathbb{F}_2 , \mathcal{X}_t has fixed intersection with V_t , so the conic passes through x . Moreover, the plane E_q has no intersection with \mathcal{X} at all, so it is mapped to a point $x_q \in X$ contained in this conic. Name the conic C_{x_q} .

Now let's find the normal bundle of the strict transform of C_4 after these two blow ups. C_4 is a complete intersection of two quadrics and its self-intersection in each of these quadrics is 8. After blowing up q , it decreases to 4. Both quadrics contain t , since their tangent plane in q coincide. Then, after the blow in t , C_4 is still a complete intersection. Its self intersection number continues to be 4, since t lied in the quadrics. Hence, after these blow ups, C_4 has normal bundle:

$$N_{C_4} = \mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$$

Now blow up C_4 , so $E_{C_4} \cong \mathbb{F}_0$. Lines of type $(0, 1)$ correspond to quadrics containing C_4 , so, in particular, $V_{C_4} \equiv (0, 1)$.

The linear system \mathcal{X} has multiplicity two in C_4 and cuts a quadric through it in $2C_4$ plus moving plane sections. So \mathcal{X}_{C_4} is a $(4, 2)$ curve.

The lines ℓ and ℓ' cut E_{C_4} in two points each, all of them lying on V_{C_4} . These are the base points of \mathcal{X}_{C_4} . So the image of E_{C_4} by σ has degree:

$$4 \cdot 2 + 2 \cdot 4 - 2 - 2 = 12$$

and it contains x (the contraction of V_{C_4}), name it S_{12}^x . It is a scroll in conics, since it maps a general line of type $(1, 0)$ to a conic through x .

To find the multiplicity of S_{12}^x in x , note that V_{C_4} has self-intersection 0 in E_{C_4} and \mathcal{X}_{C_4} intersects it in four fixed simple points, corresponding to the lines ℓ and ℓ' . Then, after the blow up at the base points, $(V_{C_4})^2 = -4$ and it is contracted to a point of multiplicity four of S_{12}^x .

To prove that S_{12}^x is also singular along the conic C_{x_q} , note that:

$$E_t \cap E_{C_4} \equiv (1, 0) + (1, 0)$$

in E_{C_4} , that is, E_t cuts E_{C_4} in two fibers (which are infinitely near if C_4 is cuspidal in q), each is mapped to a conic. But since E_t is mapped to C_{x_q} , both fibers are mapped to C_{x_q} . Therefore it is a double conic of S_{12}^x , since \mathcal{X}_{C_4} restricted to $E_t \cap E_{C_4}$ defines a 2-to-1 map.

A general quadric Q of \mathcal{Q} intersects E_{C_4} in a general line of type $(0, 1)$, which is mapped to a rational quartic curve with a double point in C_{x_q} (image of the intersection with $E_t \cap E_{C_4}$). This curve is the intersection of a quadric of \mathcal{Q}' , image of Q , with S_{12}^x . This double point is cuspidal if and only if the two intersections with $E_t \cap E_{C_4}$ are infinitely near, that is, if and only if q is cuspidal. Since E_t is mapped to C_{x_q} , this conic is described by the singular points of such quartic curves.

The four lines of type $(1, 0)$ containing the base points of \mathcal{X}_{C_4} are mapped to lines, cutting the quadrics of \mathcal{Q}' in one point each.

Since ℓ and ℓ' are mapped to the two lines through x , the four base points of \mathcal{X}_{C_4} are blown up and mapped to these two lines. Each line comes from two points, so they are double lines of S_{12}^x in S_2^x . This completes the proof. \square

A twisted cubic

Proposition 2.2.4. *A twisted cubic contained in C_4 is mapped to a degree nine scroll in conics S_9^x , having multiplicity three in x . It contains one double and one simple line through x , contained in S_2^x . It cuts the other quadrics of \mathcal{Q}' in twisted cubics. Through a general point of S_9^x there is a conic intersecting the twisted cubics and containing x . There are also three lines in S_9^x meeting the twisted cubics.*

Proof. Let C_3 be the twisted cubic. Being a rational normal curve, its normal bundle is $\mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5)$, so blowing up C_3 gives $E_{C_3} \cong \mathbb{F}_0$.

Two quadrics containing C_3 intersect each other in C_3 plus a line cutting it in two points. So, if these quadrics intersect E_{C_3} in curves of type (a, b) , it follows that:

$$2 = (a, b)^2 = 2ab$$

Hence they are of type $(1, 1)$.

If Q is a quadric containing the twisted cubic, then:

$$\mathcal{X} \cap Q \equiv 2(1, 2) + (3, 1) \equiv 2C_3 + (3, 1)$$

So the residual intersection of \mathcal{X} with Q cuts C_3 again in seven points.

Therefore \mathcal{X}_{C_3} is of type $(5, 2)$, since it cuts $(1, 1)$ -curves in seven points. It has three simple base points, corresponding to the intersections with ℓ and ℓ' , and two double points (possibly infinitely near), corresponding to the intersections with the other component (a line) of C_4 . It intersects V_{C_3} in these base points, so V_{C_3} is contracted to x .

Then \mathcal{X}_{C_3} maps E_{C_3} to a surface of degree $10 + 10 - 3 - 4 - 4 = 9$ through x , call it S_9^x . It cuts a general quadric of \mathcal{Q}' in twisted cubics, since the quadrics of \mathcal{Q} cut E_{C_3} in curves of bidegree $(1, 1)$ containing the two double points of \mathcal{X}_{C_3} . A general line of type $(1, 0)$ is mapped to a conic through x cutting each twisted cubic in one point.

The simple base points are mapped to the two lines through x in S_2^x . Since those are three points, one of them is a double line of S_9^x . The three $(1, 0)$ -lines containing these points are mapped to lines meeting the twisted cubics in \mathcal{Q}' .

Since $V_{C_3} \equiv (1, 1)$, it has self-intersection 2 in E_{C_3} . On the other hand, \mathcal{X}_{C_3} intersects this curve in its five base points. Then, after the blow up at the base points, $(V_{C_3})^2 = 2 - 5 = -3$. Therefore x is a triple point of S_9^x . \square

A conic

Proposition 2.2.5. *Let C_2 be a conic in C_4 . Then it is mapped to a weak Del Pezzo surface of degree six S_6^x , having a double point in x . It contains the two lines through x and cuts quadrics of \mathcal{Q}' in conics. There are two lines in S_6^x intersecting these conics, and each line intersects one of the lines through x .*

The \mathbb{P}^6 containing S_6^x cuts X in a tangent hyperplane section through x . It is the union of S_6^x and the image of the plane Σ containing C_2 .

Proof. As C_2 is a complete intersection, its normal bundle is:

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(4)$$

so blowing up C_2 gives $E_{C_2} \cong \mathbb{F}_2$.

If Σ is the plane containing C_2 , then $\Sigma_{C_2} \equiv e_2$. Quadrics containing C_2 intersect E_{C_2} in sections $e_2 + 2f_2$.

\mathcal{X} intersects Σ in $2C_2$ plus moving lines, so $\mathcal{X}_{C_2} \cdot e_2 = 2$ and $\mathcal{X}_{C_2} \cdot f_2 = 2$. Hence $\mathcal{X}_{C_2} \equiv 2e_2 + 6f_2$.

Remember that C_4 is of type $(2, 2)$ in V , so a conic cuts the other components in two (possibly infinitely near) points. And it cuts the lines through p in one point each. Therefore the base points of \mathcal{X}_{C_2} are two double and two simple points, all four in V_{C_2} . The double points can be infinitely near. We can also compute the degree of the image surface:

$$(2e_2 + 6f_2)^2 - 2 \cdot 4 - 2 = -8 + 24 - 8 - 2 = 6$$

So name it S_6^x (V_{C_2} is contracted to x).

Following Lemma 1.1.1, \mathcal{X}_{C_2} corresponds, in \mathbb{P}^2 , to degree six curves with two double points (from the other components of C_4), two simple points (from ℓ and ℓ'), one point of multiplicity four and one double point infinitely near to this last point.

Note that V_{C_2} corresponds to a conic with simple points in all six points above (so they are not in general position).

Performing two standard quadratic transformations, we map V_{C_2} to a point and \mathcal{X}_{C_2} to cubics with a base point, a second base point in the contraction of V_{C_2} and a third base point infinitely near to it. The result is the same if the two double points are infinitely near or not.

Hence, the conic is mapped to a weak Del Pezzo surface of degree six S_6^x , having a double point in x .

Quadrics of \mathcal{Q} intersect E_{C_2} in sections $e_2 + 2f_2$ containing the two double points, so they are mapped to conics in \mathcal{Q}' . These conics don't contain x , except for the quadric S_2^x , which intersects S_6^x in the two lines through x (images of the simple base points of \mathcal{X}_{C_2}).

The conics in \mathcal{Q}' are intersected by conics through x (image of the fibers f_2) and by two lines (image of the fibers through the simple base points).

Note that the plane Σ is mapped again to a plane. So, according to Lemma 1.3.2, Σ corresponds to the tangent hyperplane section made of S_6^x and a plane. The plane intersects S_6^x in a conic (the image of $e_2 \subset E_{C_2}$). This conic is contained in a quadric of \mathcal{Q}' : the image of the reducible quadric of \mathcal{Q} containing Σ .

□

A line

Proposition 2.2.6. *A line contained in C_4 is mapped to a scroll $S(1,2)$ through x , denoted S_3^x . The lines of its ruling are the intersections with quadrics of \mathcal{Q}' , including a line through x . Together with quartic scrolls, it forms tangent hyperplane sections of X at x .*

This cubic scroll is represented in Figure 2.3.

Proof. Let r be this line. Blowing up r gives $E_r \cong \mathbb{F}_0$, and lines of type $(0,1)$ correspond to intersections with planes through r .

A general plane containing r is cut by \mathcal{X} in r with multiplicity two and cubics with free intersections with r . So \mathcal{X}_r is of type $(3,2)$. It has two (possibly infinitely near) double points in the intersections with the other components of C_4 . It also has one base point in the intersection with one of the lines through p . These three base points lie on the curve V_r of type $(1,1)$, which is contracted to x .

In view of Lemma 1.1.1, such linear system can be birationally mapped to \mathbb{P}^2 as quintics with one triple, three double and one simple points. With two standard quadratic transformations, we get a system of conics through one point. So r is mapped to a cubic scroll $S(1,2)$, name it S_3^x . The result is the same if the two double points are infinitely near.

Looking directly in E_r , we see that the directrix line of S_3^x comes from the fibre through the simple base point of \mathcal{X}_r , while the blow up at this point is mapped to one of the lines through x . The other lines of the ruling of S_3^x come from $(1,1)$ curves through the double points, which come from the quadrics in \mathcal{Q} .

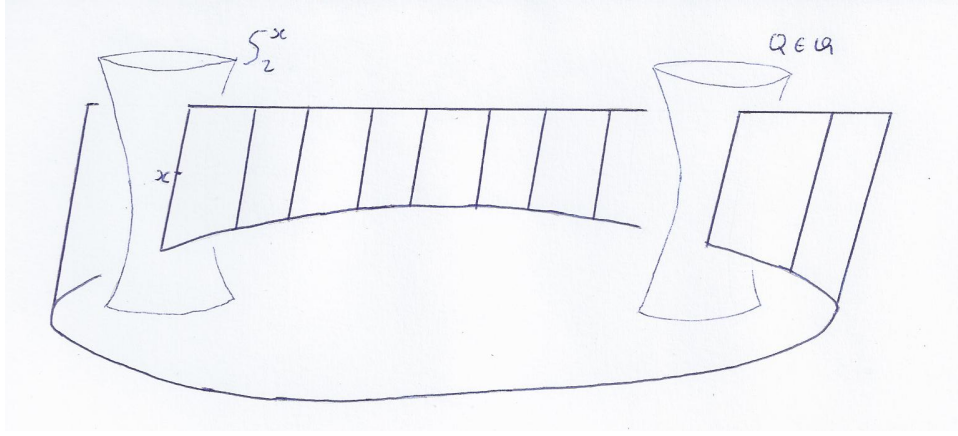


Figure 2.3: The surface S_3^x

A general plane containing r is cut by \mathcal{X} in r with multiplicity two and cubics with one double point (at the intersection with the other component of C_4 - a cubic) and one simple point (at the intersection with one of the lines through p), none of them in r . These cubics define the inverse of the tangential projection of a scroll $S(2, 2)$. So such a plane corresponds to a tangent section of X made of the cubic scroll S_3^x and a quartic scroll, both through x . They intersect each other in a twisted cubic, the image of a general $(0, 1)$ -curve in E_r . Note that special planes through r may produce scrolls of type $S(1, 3)$.

□

A double conic

Proposition 2.2.7. *Let C_2 be a double conic in C_4 , so that \mathcal{X} has multiplicity two in a second conic t infinitely near to C_2 . Then C_2 is mapped to a conic C'_2 and t is mapped to a degree six weak Del Pezzo surface S_6^x , singular at x . It contains C'_2 , being its intersection with the double plane (image of Σ) of \mathcal{Q}' .*

Proof. Let Σ the plane containing C_2 . In this situation, the Segre symbol of \mathcal{Q} is $[(111), 1]$. This means that there are two singular quadrics in \mathcal{Q} : a cone and the double plane Σ .

The linear system \mathcal{X} has multiplicity two in this conic and its tangent cone at a general point of C_2 is a double plane: the tangent plane of V at this point.

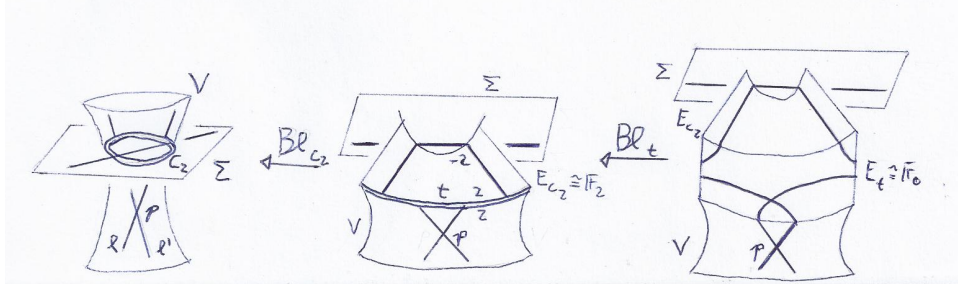


Figure 2.4: Blow up at C_2

As before, blowing up C_2 gives $E_{C_2} \cong \mathbb{F}_2$, with e_2 being the intersection with the plane Σ . Again, $t = V_{C_2}$ is a section of type $e_2 + 2f_2$ in E_{C_2} , and \mathcal{X}_{C_2} has type $2e_2 + 6f_2$. But now, the tangency condition implies that in each fibre of E_{C_2} , \mathcal{X}_{C_2} has a double point at the intersection of t with this fibre. Hence:

$$\mathcal{X}_{C_2} \equiv 2e_2 + 6f_2 \equiv 2t + \{2f_2\}$$

So it maps E_{C_2} to a conic $C'_2 \subset X$. Note that t is the other component of C_4 , but now it is infinitely near to C_2 .

Now blow up t . It's a complete intersection of V and E_{C_2} . In V , $t^2 = (C_2)^2 = 2$; in E_{C_2} , $t^2 = (e_2 + 2f_2)^2 = 2$. Hence its normal bundle is:

$$N_t = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

and $E_t \cong \mathbb{F}_0$.

Both E_{C_2} and V intersect E_t in $(0,1)$ lines. Since the moving part of \mathcal{X}_{C_2} intersected t in two points, it follows that \mathcal{X}_t is made of curves of type $(2,2)$. It has two simple base points at the intersections with ℓ and ℓ' .

In Figure 2.4 a sketch of the blow up at C_2 is given.

\mathcal{X}_t is birationally equivalent to quartic curves in \mathbb{P}^2 with two double points and two simple points. Here, V_t is a line through one of the double points and both simple points. With a quadratic standard transformation, one maps \mathcal{X}_t to plane cubic with two base points and a third one infinitely near to one of them (the image of V_t).

This is the same linear system found in Proposition 2.2.5. Therefore E_t is mapped to S_6^x , a degree six weak Del Pezzo surface with a double point in x .

Since $\Sigma_{C_2} \equiv e_2$, this plane has no intersection with E_t . With the contraction of E_{C_2} to C'_2 , the image of Σ intersects S_6^x in C'_2 .

□

In Section 2.2.3 it will be proven that C'_2 is a double conic of X .

A double line

Proposition 2.2.8. *Let r be a double line of C_4 , so \mathcal{X} has multiplicity two in a second line t infinitely near to r . Then t is mapped to the surface $S_3^x \cong S(1, 2)$, defined in Proposition 2.2.6. The image of r is a line r' of the ruling of S_3^x .*

Proof. As in the previous case, the tangent cone of \mathcal{X} at a general point of r is the tangent plane of V at this point, with multiplicity two.

If $r \equiv (1, 0)$ in V , the other components of C_4 are a $(1, 0)$ -line infinitely near to r and two other lines of type $(0, 1)$.

As before, blowing up r gives $E_r \cong \mathbb{F}_0$. \mathcal{X}_r has degree $(3, 2)$ and has two double points (which can be infinitely near) and one simple base point. But in each line $(1, 0)$ the system \mathcal{X}_r has a double point in the intersection of this line with $t = V_r \equiv (1, 1)$. So:

$$\mathcal{X}_r \equiv (3, 2) \equiv 2t + (1, 0)$$

and r is mapped back to a line $r' \subset X$. Note that t is a line of V infinitely near to r .

Since t is a complete intersection of V and E_r , and since $t^2 = 0$ in V and $t^2 = 2$ in E_r , its normal bundle is:

$$N_t = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

so blowing up t gives $E_t \cong \mathbb{F}_2$. The system \mathcal{X}_t cuts $E_r \cap E_t \equiv e_2$ in one moving point and each fiber in two points. Hence:

$$\mathcal{X}_t \equiv 2e_2 + 5f_2$$

It has two double points (in the intersection with the other components of C_4) and one simple point (in the intersection with a line through p). All of them lie on $V_t \equiv e_2 + 2f_2$.

The blow up at r is represented in Figure 2.5.

Using Lemma 1.1.1, \mathcal{X}_t is mapped, in \mathbb{P}^2 , to curves of degree five with two double points, one simple point, one point with multiplicity three and one double point infinitely near to this point.

Moreover, $V_t \equiv e_2 + 2f_2$ is mapped to a conic through all those five points. After two standard quadratic transformations, \mathcal{X}_t is a system of conics with

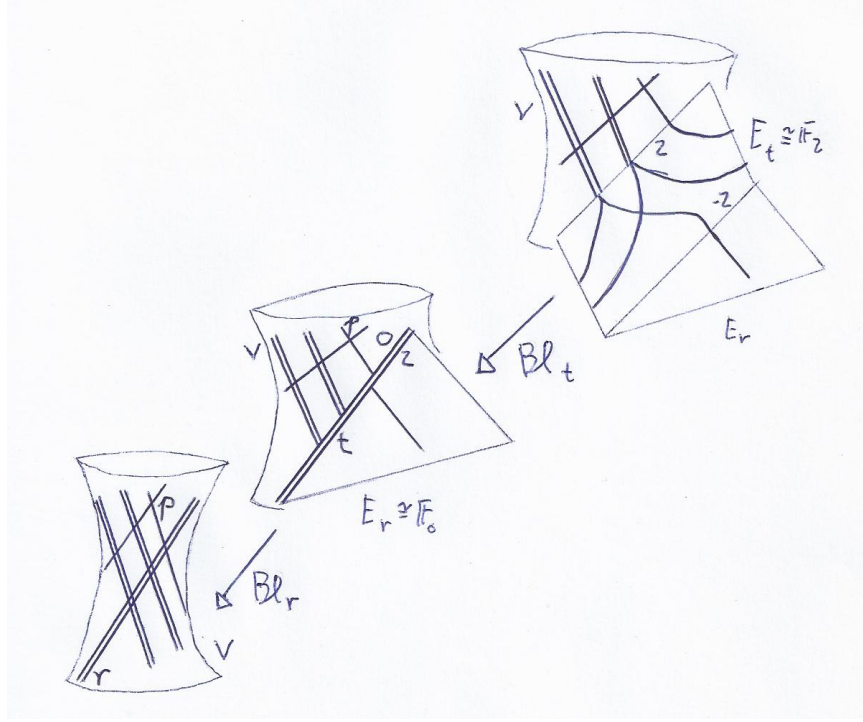


Figure 2.5: Blow up at r

one base point, while V_t is another point in \mathbb{P}^2 . Then E_t is mapped to a cubic scroll S_3^x through x .

In E_t , the sections of type $e_2 + 2f_2$ containing the two double points are mapped to lines of the ruling of S_3^x . This includes the reducible section consisting of the union of $e_2 = E_r \cap E_t$ with the two fibers through the double points. In particular, r' is a line of the ruling of S_3^x .

□

2.2.3 Singularities of X

In this section we will study the singularities of X . The map σ is an isomorphism outside V , which is contracted to x , so these singular points lie on the images of exceptional divisors. By Lemma 2.2.2, only C_4 can produce singularities. Moreover:

Lemma 2.2.9. *A singularity of X is the image of a singular point of C_4 . This includes points in non reduced components of C_4 .*

Proof. Since x is a general point in X , Terracini's Lemma implies that an isolated singular point does not lie on $T_x X$. Then it is projected by τ to a point q . As just noted above, $q \in C_4$.

But singular points of X are contractions, via σ , of surfaces (or curves) which have fixed intersection with the strict transform of \mathcal{X} after the blow up at its base locus.

If q is a smooth point in C_4 , the tangent cone of \mathcal{X} at q is a pair of moving planes containing the tangent line of C_4 in q . So after the blow up at q , σ maps E_q to a conic. Therefore q is not the projection of an isolated singular point of X .

Then q is a singular point of C_4 .

On the other hand, if the singularity lies on a singular curve of X , this curve is the contraction of an exceptional divisor of the blow of \mathcal{X} in its base locus. By Lemma 2.2.2, this exceptional divisor comes from C_4 . And by the analysis made in Section 2.2.2 the associated component of C_4 is either a double line or a double conic. In both cases, a point in a singular curve of X is mapped to a singular point of C_4 . □

Note that if q is the projection of a singular point of X , the tangent cone of \mathcal{X} in q is a fixed double plane: the tangent plane of V at q . After the blow up at q , \mathcal{X} cuts the exceptional plane E_q in a fixed double line, so this plane is re-contracted to a point $x_q \in X$.

By the above Lemma and Proposition 2.1.6, it follows:

Corollary 2.2.10. *Each singularity of X is also a singularity of a quadric of \mathcal{Q}' corresponding to a root of multiplicity greater than 1 in the Segre symbol.*

The multiplicity of X in x_q can be easily computed using Lemma 1.3.3.

Lemma 2.2.11. *Any singular point of X is of multiplicity two.*

Proof. Let x_q be a singular point of X . Then by Lemma 2.2.9, x_q is the image via σ of a singular point q of C_4 . More precisely, after the blow up at q , \mathcal{X} intersects E_q in a fixed double line, namely V_q . The plane E_q is then contracted to x_q . Conversely, the preimage of x_q via σ is the point q alone.

Let Ω be a general plane through q . Then $\mathcal{X} \cap \Omega$ has multiplicity two in q . Blowing up q , $\mathcal{X} \cap \Omega$ intersects the exceptional curve $e_q = E_q \cap \Omega$ in a fixed double point. Blowing up this point, we get $(e_q)^2 = -2$ and e_q has no intersection with $\mathcal{X} \cap \Omega$. Then it is contracted to a double point, namely

x_q , of $\sigma(\Omega)$. By Lemma 1.3.3, x_q is a double point of X . □

In order to understand better the singularity of X at x_q , we will use the fact that the blow up of \mathbb{P}^3 along a singular curve is a singular threefold. Then one can understand the singularities of X by considering the singularities of the blow up of \mathbb{P}^3 along C_4 .

By studying the different possible singular points q of C_4 , we blow up \mathbb{C}^3 along a curve having the same type of singularity in the origin. Then this blow up gives a singular threefold that represents locally the singularity of X in x_q .

We will now consider the possible singularities of C_4 .

A transversal intersection of two simple branches of C_4 : In this case, q is a nodal point of C_4 . Then one needs to analyse the blow up of \mathbb{C}^3 along a curve with a nodal point in the origin. For instance, consider the curve $xy = z = 0$. The blown up threefold is the subvariety of $\mathbb{C}^3 \times \mathbb{P}^1$ with coordinates $(x, y, z), (u : v)$ given by the equation $uz - xyv = 0$. In one of the affine charts $\mathbb{C}^4 = \mathbb{C}^3 \times \mathbb{C}$, it is given by $z - xyv = 0$ and it is smooth. In the other affine chart, its equation is $uz - xy = 0$, which is a rank four quadric cone in \mathbb{C}^4 . Therefore this is the local equation of the singularity $x_q \in X$.

A cuspidal point: This case can be explained by the blow up of \mathbb{C}^3 along the curve $z = x^3 - y^2 = 0$. This produces the threefold $uz - v(x^3 - y^2) = 0$ in $\mathbb{C}^3 \times \mathbb{P}^1$. Its singularity in $(0, 0, 0), (0 : 1)$ is given locally by $uz - x^3 + y^2 = 0$. Then the tangent cone is $uz + y^2 = 0$, which has rank 3.

Blowing up $uz - x^3 + y^2 = 0$ in the origin, one sees that there is no singular point infinitely near to it.

A contact of two branches of C_4 : A local representation of this singularity is given by the curve $z = x(x - y^2) = 0$. The blow up of \mathbb{C}^3 along this curve produces the threefold $uz - vx(x - y^2) = 0$ having a double point in $(0, 0, 0), (0 : 1)$ given locally by $uz - x^2 + xy^2 = 0$. The tangent cone is $uz - x^2 = 0$, which also has rank 3.

Blowing up the origin, one of the affine charts is given by:

$$x = \mu y \quad z = \nu y \quad u = \rho y$$

In this chart, the strict transform of $uz - x^2 + xy^2 = 0$ is $\nu\rho - \mu^2 + \mu y = 0$, which has a double point at the origin. In the other charts there are no further singular points.

Therefore, x_q is a double point with a second double point infinitely near to it.

A general point of a double curve: This singularity can be explained by the blow up of \mathbb{C}^3 along the curve $z = x^2 = 0$, which gives $uz - x^2v = 0$. The double point is given locally by $uz - x^2 = 0$. This is also the equation of the tangent cone, which has rank 3, that is, it has a double line.

As remarked in Proposition 2.2.7 and in Proposition 2.2.8, a double line of C_4 is mapped to a line in X , and a double conic is mapped to a conic in X . Then the double point actually moves in a double curve of X .

A transversal intersection of three branches of C_4 : Locally, this singular point is given by $z = xy(x+y) = 0$. Blowing up \mathbb{C}^3 along this curve gives a threefold with a double point locally given by $uz - xy(x+y) = 0$. The tangent cone is $uz = 0$, that is, a pair of three-dimensional planes.

Now, blow up the origin and consider the affine chart given by:

$$y = \eta x \quad z = \nu x \quad u = \rho x$$

The strict transform of $uz - xy(x+y) = 0$ is $\rho\nu - x\eta - x\eta^2 = 0$. It has double points in $(x, \eta, \nu, \rho) = (0, 0, 0, 0)$ and $(0, 1, 0, 0)$. There is a third double point in the origin of the chart given by:

$$x = \mu y \quad z = \nu y \quad u = \rho y$$

The three double points are collinear. They lie on the intersection of the exceptional divisor with the strict transform of the plane given by $z = u = 0$.

Therefore X has a double point in x_q with three collinear double points infinitely near to it.

A transversal intersection of a double and a simple branch of C_4 : This singularity can be locally given by $z = x^2y = 0$. Blowing up \mathbb{C}^3 along this curve, gives $uz - x^2y = 0$, which has a double line through the origin. Moreover, blowing up the origin, in the chart given by:

$$y = \eta x \quad z = \nu x \quad u = \rho x$$

its strict transform is $\nu\rho - \eta x = 0$. It has a double point in the origin. This point does not lie on the double line, which cannot be seen in this chart.

Therefore, X has a double line through x_q and a double point infinitely near to x_q .

A transversal intersection of two double branches of C_4 : Note that this is precisely the case in which C_4 is two double lines. As already noted, a double line of C_4 is mapped to a double line of X . Then x_q lies on the intersection of two double lines of X .

This can also be verified blowing up $z = x^2y^2 = 0$, giving $uz - x^2y^2 = 0$. This threefold has two double lines intersecting in the origin. The tangent cone is a pair of three-dimensional planes.

The main results of this section are now collected:

Proposition 2.2.12. *A singular point of C_4 is mapped to a double point of X . This includes double components of C_4 , which are mapped to double curves of X . More specifically let q be a singular point of C_4 and x_q its image. Then the tangent cone of X in x_q has:*

- rank 4, if q is the transversal intersection of two simple branches of C_4 ;
- rank 3, if q is a cuspidal point, a point of contact of two simple branches or a general point of a double component of C_4 ;
- rank 2, that is, it is a pair of three-dimensional planes, if q is the intersection of three simple branches, the intersection of a double and a simple branch or the intersection of two double branches of C_4 .

Moreover, x_q is a singular point of a quadric of \mathcal{Q}' corresponding to a root of multiplicity greater than one in the Segre symbol.

2.2.4 Summing up

Now that we have enough information, we can easily describe the possible varieties X . We just consider different possibilities for a curve C_4 of type $(2, 2)$ in V , which are:

- (E1) a smooth elliptic quartic (general case) – $(2, 2)$
- (E2) an irreducible rational nodal quartic – $(2, 2)$
- (E3) an irreducible rational cuspidal quartic – $(2, 2)$

- (E4) a twisted cubic and a transversal line $-(2, 1) + (0, 1)$
- (E5) a twisted cubic and a tangent line $-(2, 1) + (0, 1)$
- (E6) two transversal conics $-(1, 1) + (1, 1)$
- (E7) two tangent conics $-(1, 1) + (1, 1)$
- (E8) a conic and two lines intersecting it in two points $-(1, 1) + (1, 0) + (0, 1)$
- (E9) a conic and two lines intersecting it in one point $-(1, 1) + (1, 0) + (0, 1)$
- (E10) four lines $-(1, 0) + (1, 0) + (0, 1) + (0, 1)$
- (E11) one double and two simple lines $-2(1, 0) + (0, 1) + (0, 1)$
- (E12) two double lines $-2(1, 0) + 2(0, 1)$
- (E13) one double conic $-2(1, 1)$

We will refer to the resulting Bronowski variety by the string on the left in the above list. This leads to the result:

Theorem 2.2.13. *Let X be a Bronowski threefold satisfying hypothesis (H) and with fundamental surface being a smooth quadric. Then X corresponds to one of the 13 quartic curves listed above.*

Note that in all cases, C_4 is mapped to a (possibly reducible) scroll in conics with degree 12 and multiplicity four in x (if there is a double curve in C_4 its image is counted with multiplicity two). This scroll cuts each quadric of \mathcal{Q}' in curves of the same type of C_4 . The only exception is the quadric through x, S_2^x , which is cut in two lines, which are double lines of the degree 12 scroll. Since the singular points of X come from C_4 , the scroll contains these points.

A description of these threefolds is given in Theorem 2.4.1, together with those obtained in the case in which V is a cone.

2.3 The singular case

Suppose now that V is a quadric cone with vertex q . According to Lemma 2.1.1, either $\text{mult}_q \mathcal{X} = 4$ and all quadrics of \mathcal{Q} are singular in q or

$\text{mult}_q \mathcal{X} = 3$ and there is a quadric in \mathcal{Q} which is a smooth in this point. In both cases, we have:

$$\mathcal{X} \cap V = 2C_4 + 2\ell$$

where C_4 is the base locus of \mathcal{Q} and ℓ is the line joining p and q . Note that ℓ is not a double curve of \mathcal{X} . The multiplicity 2 in the intersection is due to a tangency of \mathcal{X} and V , that is, there is a line ℓ' infinitely near to ℓ in the base locus of \mathcal{X} .

As it was remarked in Lemma 2.1.2, the point p is mapped to a quadric cone S_2^x through the smooth point x . It belongs to the family \mathcal{Q}' .

Made these considerations, we analyse the two possibilities.

2.3.1 When the multiplicity in q is three

If $\text{mult}_q \mathcal{X} = 3$, there are two possibilities. Either \mathcal{Q} has a smooth quadric or it is a pencil of cones with moving vertex.

In the first case, the general quadric in \mathcal{Q}' is smooth, so this family has at most four cones. Then it seems that x is not a general point of X , since it lies on one of the four cones of \mathcal{Q}' . In fact:

Lemma 2.3.1. *Suppose V is a quadric cone with vertex q and suppose there is a smooth quadric in \mathcal{Q} . Then X is projectively equivalent to one of the threefolds described in Section 2.2.*

Proof. The idea is to change coordinates to fit this situation in the smooth case of Section 2.2. This is done using the Cremona transformation of Section 2.1.3. Choose a point \bar{p} in \mathbb{P}^3 such that the quadric of \mathcal{Q} containing it is smooth. Apply the Cremona transformation f defined in Section 2.1.3 and then apply the inverse transformation associated to the point $f(\bar{p})$ (instead of using the point $p' = f(V)$).

The image of \mathcal{X} by the composition of these two maps is a similar linear system having fixed intersection with a smooth quadric, instead of a cone. The quartic curve in its base locus is of the same type of the original C_4 . \square

As remarked in [HP, p.305], in the second case there is only one possibility, in which \mathcal{Q} is a pencil of cones tangent to a plane Π along a fixed line r . The base locus of \mathcal{Q} is $C_4 = 2r + C$, where C is a conic intersecting r in one point. This implies that the only reducible quadric in \mathcal{Q} is the union of Π and the plane Σ containing C .

Note that, since \mathcal{X} intersects a cone Q of \mathcal{Q} different from V in the union of C_4 and moving plane sections, such quadric is mapped back to a cone, and the vertex of this cone is the image of the vertex of Q .

This configuration produces a new Bronowski threefold, which we identify as (E14):

Proposition 2.3.2. *Suppose \mathcal{Q} is a pencil of cones with moving vertex and let q be the vertex of V . Then the base locus of \mathcal{Q} is $2r + C$, where r is a line and C is a conic, and the multiplicity of \mathcal{X} in q is three.*

The threefold X has a double conic L , the image of r . This conic is described by the vertices of cones of the one-dimensional family \mathcal{Q}' contained in X . There is exactly one reducible quadric in this family, which intersects L in one point.

The line r' in the base locus of \mathcal{Q} which is infinitely near to r is mapped to a cubic scroll. The conic C is mapped to a weak Del Pezzo surface of degree six having a double point in x .

Proof. The first part has already been explained.

By Lemma 2.1.1, \mathcal{X} has multiplicity two in $p \notin r$, C , r and in a line r' infinitely near to r , corresponding to the plane Π . Then:

$$\mathcal{X} \cap \Pi = 4r + \{\text{lines}\}$$

The linear system also contains the line ℓ through p and q and a line ℓ' infinitely near to ℓ .

The blow ups that now follow are represented in Figure 2.6.

Start blowing up q . The degree three curve \mathcal{X}_q has a double point in r_q and a second double point in r'_q infinitely near to it. It also has a simple point in ℓ_q and a second simple point in ℓ'_q infinitely near to it. Therefore, the line $t = \Pi_q$ through the two double points is a fixed component of \mathcal{X}_q . It is the tangent line to the conic V_q in r_q . Hence:

$$\mathcal{X}_q = t + \{\text{conics}\}$$

where the moving conics have four simple base points, that is, they are tangent to V_q in ℓ_q and r_q . This moving part is equivalent, by a standard quadratic transformation in E_q , to lines through a point. Then E_q is mapped to a line. Blowing up ℓ and ℓ' , it follows that the image of E_q is the line through x .

Let Π' be a general plane containing r , so that $r = \Pi \cap \Pi'$. After the blow up at q , we have that $r^2 = 0$ in both planes. Then its normal bundle is:

$$N_r = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

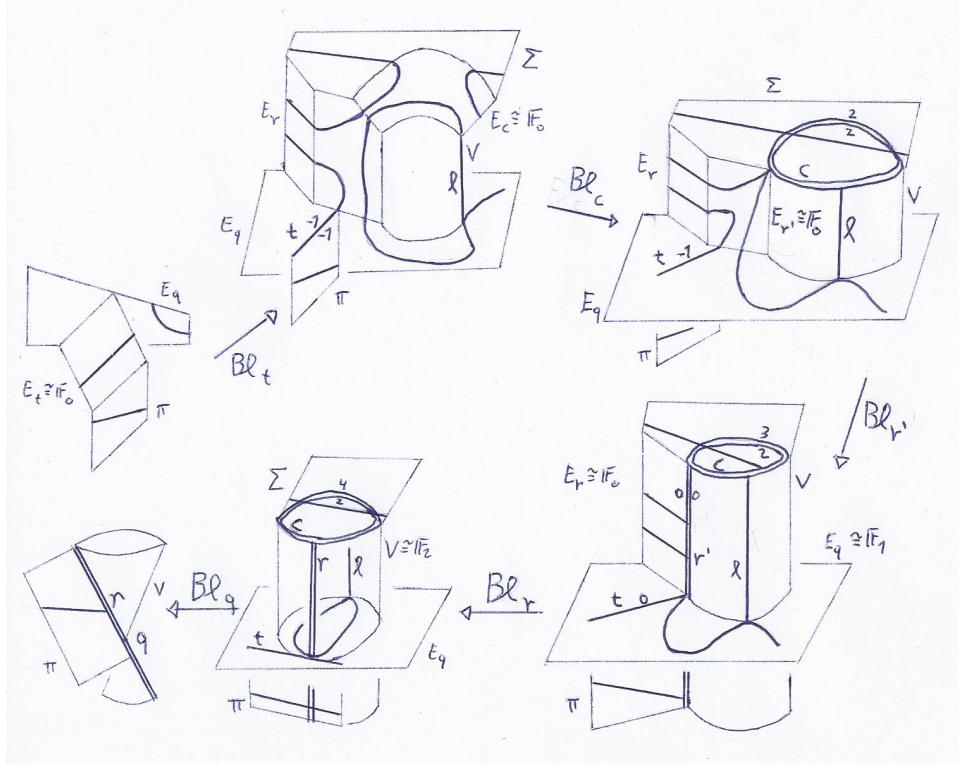


Figure 2.6: \mathcal{X} and the blow ups in Proposition 2.3.2

Now blow up r . Before the blow ups, the intersection $\mathcal{X} \cap \Pi'$ consisted of r with multiplicity two and cubic curves. These curves intersected r in q and two moving points. Then, after the blow ups at q and r , $\mathcal{X}_r \equiv (2, 2)$. But in $E_r \cong \mathbb{F}_0$, the line $r' = V_r$ is a double curve of \mathcal{X}_r . Therefore:

$$\mathcal{X}_r \equiv (2, 2) = 2r' + (2, 0)$$

The moving part consists of pairs of fibers over points of r , mapping E_r to a conic L . Since r is the locus of vertices of cones in \mathcal{Q} , L is the locus of vertices of cones in \mathcal{Q}' .

In E_q , both t and the moving conics intersect E_r in r' .

The line $r' = E_r \cap V$ has self-intersection 0 in E_r and 0 in $V \cong \mathbb{F}_2$. Then its normal bundle is:

$$N_{r'} = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

Blow up r' , giving again $\mathcal{X}_{r'} \equiv (2, 2)$. This linear system has a double point in $C_{r'}$, since C intersected r in one point and since C is tangent to

II. It also has a simple point in $t_{r'}$. By Lemma 1.1.1, $\mathcal{X}_{r'}$ corresponds to a linear system of quartic curves in \mathbb{P}^2 with three double points and one simple point. After a quadratic standard transformation, we get conics with one base point. Therefore $E_{r'}$ is mapped to a cubic scroll.

As it was already noted before, there is only one reducible quadric in \mathcal{Q} , namely the union of Π and Σ . The linear system \mathcal{X} intersects Π in $4r$ and moving lines, mapping it to a plane, and intersects Σ in $2C$ plus moving lines, mapping it to a plane too. Therefore the image of this pair of planes is a reducible quadric in \mathcal{Q}' .

The plane Σ intersects $E_{r'}$ in the fiber through the double point of $\mathcal{X}_{r'}$, and $\Pi_{r'} \equiv (0, 1)$ contains the simple point. Since $(E_r)_{r'} \equiv (0, 1)$ contains none of these base points, the image of $\Sigma \cup \Pi$ intersects L in one point.

Next, we investigate the image of C . Before the blow ups, the conic C was the complete intersection of Σ and V . And this is still the case after the blow ups, since r and r' lied in $V \setminus \Sigma$ and $q \notin \Sigma$ (there is no conic in V through q , so $q \notin C$). In Σ , C had two points blown up, so $C^2 = 2$. In V , none of the blow ups has affected its self-intersection, so $C^2 = 2$. Then:

$$N_C = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

Since $\mathcal{X} \cap \Sigma = 2C + \{\text{lines}\}$, the blow up at C gives $\mathcal{X}_C \equiv (2, 2)$. It has a base point in ℓ_C and a second base point infinitely near to it. Both points lie on $V_C \equiv (0, 1)$. The linear system \mathcal{X}_C corresponds, in \mathbb{P}^2 , to quartics with two double points and two simple points. A standard quadratic transformation maps it to cubics with three base points. These points lie on the image of V_C , which is a line. Hence E_C is mapped to a weak Del Pezzo sextic surface having a double point in x .

The line t is the complete intersection of E_q and Π . In E_q , $t^2 = -1$, since both r and r' intersected E_q in points of t . In Π , t was the exceptional curve of the blow up at q . Since both r and r' lied in Π , it follows that $t^2 = -1$. Then, the normal bundle of t is:

$$N_t = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

In both Π and E_q , the moving part of \mathcal{X} does not intersect t . Then blowing up t gives $\mathcal{X}_t \equiv (0, 1)$ with no base points, and E_t is mapped to a line. This is the directrix line of the cubic scroll, since E_t intersects $E_{r'}$ in the exceptional divisor of the blow up at the simple base point of $\mathcal{X}_{r'}$.

This completes the analysis of the images of the base curves of \mathcal{X} . We now proceed to the study of the singularities of X . According to Lemma 2.2.9 (which applies with no change), these singularities lie on L , the image

of r . Let q' be a general point of r and let Ω be a general plane through q' . Then $\mathcal{X} \cap \Omega$ consists of degree five curves with a double point in q' , a second double point infinitely near to it (corresponding to r'), two double points in $C \cap \Omega$, a simple point in $l \cap \Omega$ and a second simple point infinitely near to it.

Blowing up q' , $\mathcal{X} \cap \Omega$ intersects the exceptional curve e in a fixed double point. Blowing up this point, e no longer intersects $\mathcal{X} \cap \Omega$ and $e^2 = -2$. Then it is mapped to a double point of the image of Ω . By Lemma 1.3.3, it is a double point of X . Therefore, L is a double conic of X . □

2.3.2 When the multiplicity in q is four

This implies that C_4 is a union of four lines through q , the intersection of V with another cone with vertex q . Since all quadrics of \mathcal{Q}' are singular and the general one is a cone, there is a curve in X described by these vertices.

Lemma 2.3.3. *Blowing up the vertex q of V , the plane E_q is mapped to the line L described by the vertices of the cones of \mathcal{Q}' . It is a double line of X . The conic $C = V \cap E_q$ is mapped to the line through x in the cone S_2^x .*

Proof. The following blow ups are represented in Figure 2.7.

Consider the blow up at q . Then \mathcal{X}_q is made of quartic curves having four double and two simple points. The simple points are infinitely near. These six points lie on the conic $C = V_q$. So the linear system \mathcal{X}_q has a fixed conic C and then conics through four base points. This moving part is Cremona equivalent to lines through a point. It maps E_q to a line, name it L .

In E_q , each conic through the four base points is mapped to a point of L . So these points are double points of X . These moving conics in E_q are also the intersections of this plane with the quadrics of \mathcal{Q} , which are also singular in q . Hence, L is the line described by the vertices of the quadrics \mathcal{Q}' .

Before blowing up the conic C , one should blow up the four (possibly infinitely near) double lines of \mathcal{X} , that is, C_4 , in order to avoid other fixed components. After this blow up, C continues to be a complete intersection of V and E_q . But now, its self intersection in E_q has decreased by four, that is, $C^2 = 0$ in E_q . In V , $C^2 = -2$, as before. So:

$$N_C = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

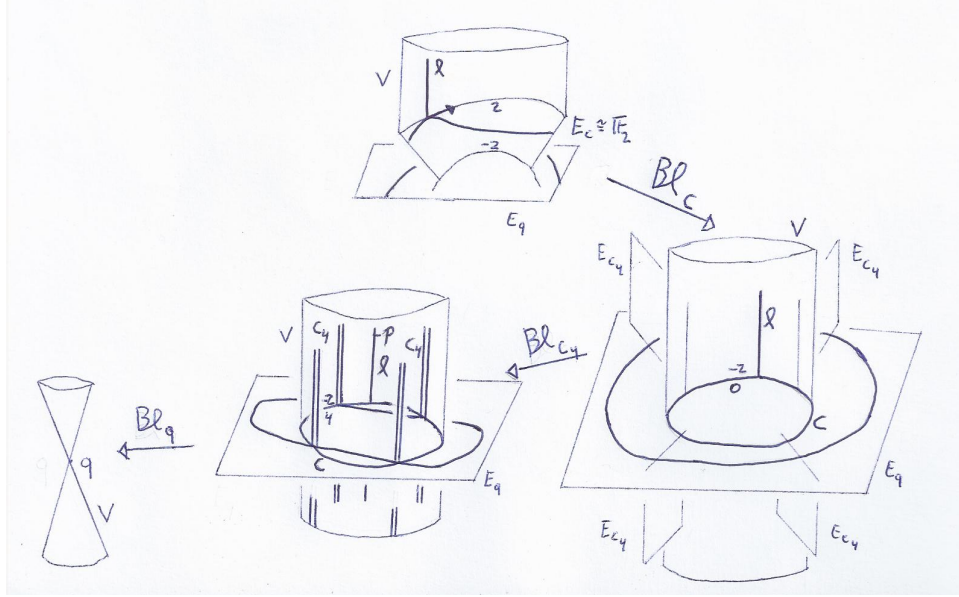


Figure 2.7: Blow up at q and C

Note that, before the blow ups, \mathcal{X}_q intersected C only in the four points coming from the lines of C_4 . So now, \mathcal{X}_C does not intersect $E_q \cap E_C$, which is the (-2) -section of $E_C \cong \mathbb{F}_2$. Therefore $\mathcal{X}_C \equiv e_2 + 2f_2$.

This linear system has two infinitely near base points, in the intersection with the line ℓ . These two points lie on V_C , which is also a section of type $e_2 + 2f_2$. Hence, V_C is contracted to the point x .

By Lemma 1.1.1, \mathcal{X}_C corresponds in \mathbb{P}^2 to conics with four base points in two infinitely near pairs. After a standard quadratic map, we get a pencil of lines through a point. So E_C is mapped to a line. It is the one line through x of the cone S_2^x .

□

Next, we will find the image of ℓ .

Lemma 2.3.4. *The lines ℓ' and ℓ are mapped respectively to the line through x in S_2^x and to the vertex of this cone.*

Proof. The linear system \mathcal{X} cuts a general plane Π through ℓ in this line plus degree four curves with multiplicity three in q and passing through p . There is also another base point infinitely near to q , which will now be described. Consider the blow up at q . Remember that $C = V_q$ is a fixed part of \mathcal{X}_q . It

intersects the line Π_q in two points, one of them in ℓ . The other point is the infinitely near point mentioned above.

After this blow up, the line ℓ has normal bundle:

$$N_\ell = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

Remember that \mathcal{X} contains a line ℓ' infinitely near to ℓ , corresponding to the tangent plane of V at a general point of ℓ . So ℓ' is a $(0,1)$ -curve in E_ℓ , intersecting E_q in a point of the conic C .

Since the degree four curves in Π intersected ℓ in p , it follows that:

$$\mathcal{X}_\ell \equiv (1,1) \equiv \ell' + F_p$$

where F_p is the fiber over the point $p \in \ell$.

The line ℓ' is the intersection of $V \cong \mathbb{F}_2$ (since it was blown up at q) and E_ℓ . So:

$$N_{\ell'} = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

In $E_{\ell'}$, E_q intersects in a line of type $(1,0)$, while V and E_ℓ in lines of type $(0,1)$. $\mathcal{X}_{\ell'}$ has a base point in the intersection with $C = E_q \cap V$ and another in the intersection with $F_p \subset E_\ell$.

Therefore, $\mathcal{X}_{\ell'}$ is made of curves of type $(1,1)$ with two simple base points. It maps $E_{\ell'}$ to a line through x (the contraction of $V_{\ell'}$).

The surface E_ℓ is mapped to a point. It is the intersection of the line through x and L , that is, the vertex of S_2^x .

These blow ups are sketched in Figure 2.8.

□

Note that there is no contradiction with Lemma 2.3.3, since, in E_C , the blow up at the base points determined by ℓ' and ℓ are mapped to the line through x and the vertex of S_2^x .

Before entering into the specific cases, let's make a remark on Segre symbols. Since \mathcal{Q} has no smooth element, the Segre symbol is not defined. But a general plane section of \mathcal{Q} gives a pencil of conics. Since V is irreducible, there is a smooth conic in this pencil. Moreover, the singular elements in this pencil correspond to the special elements (i.e. pairs of planes and double planes) of \mathcal{Q} .

So I will denote the *Segre symbol* of \mathcal{Q} to be the Segre symbol of the pencil of conics defined by a general plane section of \mathcal{Q} . To distinguish from the other notation, I will use two brackets. For example, the special elements of the pencil with symbol $[[1, 2]]$ are two pairs of planes, while the one with symbol $[[(21)]]$ has only a double plane.

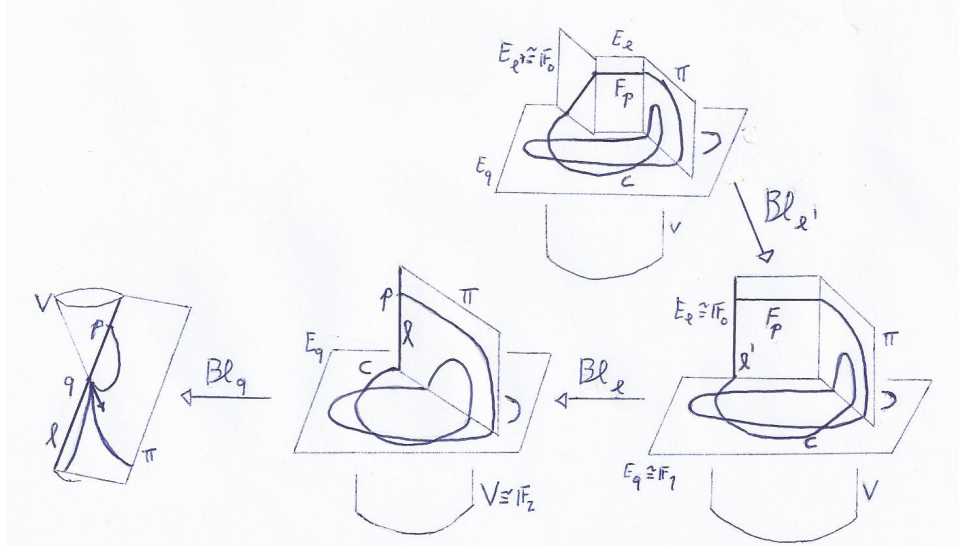


Figure 2.8: Blow up at ℓ and ℓ'

There are five possibilities for C_4 , which give five possibilities for the Segre symbol of \mathcal{Q} :

- (E15) four distinct lines – $[[1, 1, 1]]$
- (E16) one double and two simple lines – $[[2, 1]]$
- (E17) two double lines – $[[(11), 1]]$
- (E18) a triple and a simple line – $[[3]]$
- (E19) a line with multiplicity four – $[[(21)]]$

(E15) Four simple lines

Let r be a simple line in C_4 . Then:

Lemma 2.3.5. *A simple line $r \subset C_4$ is mapped to a cubic scroll S_3^x through x . Its directrix line is L and the cones of \mathcal{Q}' intersect this scroll in lines of the ruling. Figure 2.9 gives a representation of S_3^x .*

Proof. The linear system \mathcal{X} cuts a general plane Π through r in $2r$ plus cubics with a double point at q . Consider the blow up at q , as done in Lemma 2.3.3. The line r intersects E_q in a point of the conic C and in a

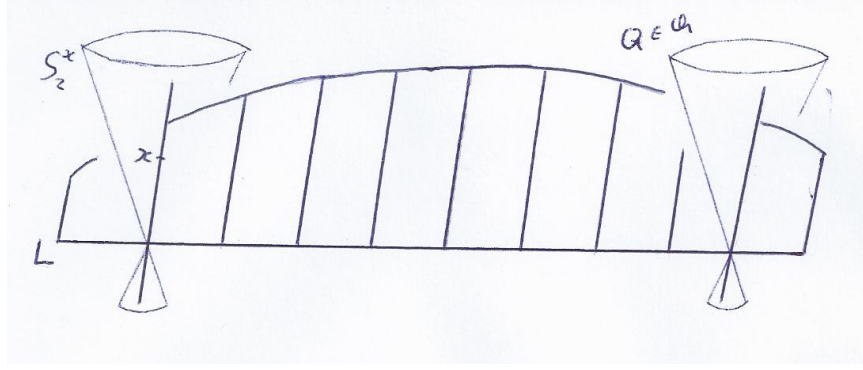


Figure 2.9: The scroll S_3^x

point of the moving part of \mathcal{X}_q . Since it is the complete intersection of two planes (blown up at q), its normal bundle is:

$$N_r = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

Now blow up r . \mathcal{X}_r cuts $E_q \cap E_r \equiv (1, 0)$ in two points. One of them lies on C , so it is a fixed point. The other is a moving point. Moreover \mathcal{X}_r cuts lines $\Pi_r \equiv (0, 1)$ in one moving point.

Therefore, \mathcal{X}_r is made of $(1, 2)$ curves with one base point. These correspond to cubics in \mathbb{P}^2 with one double and two simple points, which are Cremona equivalent to conics with one base point. Hence E_r is mapped to a cubic scroll $S(1, 2)$. Name it S_3^x , since $V_r \equiv (0, 1)$ is contracted to x .

The other quadrics of \mathcal{Q} also intersect E_r in curves of type $(0, 1)$. Then the quadrics of \mathcal{Q}' intersect S_3^x in lines of its ruling. Since E_q intersects E_r in the line $(1, 0)$ through the base point, it is mapped to the directrix line of S_3^x .

□

There are four cubic scrolls through the point x , which are contracted by the tangential projection τ to lines. In all four of these scrolls, the directrix line is L .

Note that, blowing up all four double lines of \mathcal{X} , the four exceptional divisors do not intersect. Therefore, the only curves in common to the four scrolls come from contractions of surfaces. So each scroll intersects a cone of \mathcal{Q}' in a different line. The only exception is the cone S_2^x through x , which is intersected by all four in the line through x .

Since x is a general point in X , there are cubic scrolls through other general points of S . The following Lemma describes the image, via τ , of these surfaces:

Lemma 2.3.6. *Through a general point $y \in X$ there are four scrolls $S(1, 2)$ contained in X intersecting quadrics of \mathcal{Q}' in lines and having directrix line L . The image via τ of these scrolls in \mathbb{P}^3 are quadric cones through $\tau(y)$ containing three of the four double lines of \mathcal{X} and ℓ .*

Proof. Let r_1, r_2, r_3 be three of the four double lines of \mathcal{X} . Let S be a quadric surface containing these lines and ℓ . This implies that S is a cone with vertex q . Consider the blow up at q , so that $S \cong \mathbb{F}_2$. Then:

$$\mathcal{X} \cap S \equiv 5e_2 + 10f_2 \equiv 4e_2 + 2r_1 + 2r_2 + 2r_3 + \ell + \{e_2 + 3f_2\}$$

where the moving part has a base point in p , which is a double point of \mathcal{X} .

Using Lemma 1.1.1, the moving part is birationally equivalent to cubics in \mathbb{P}^2 with one simple point, one double point and a third simple point infinitely near to it. Therefore S is mapped to a cubic scroll $S(1, 2)$.

The intersection with quadrics of \mathcal{Q} is:

$$\mathcal{Q} \cap S \equiv 2e_2 + 4f_2 \equiv 2e_2 + r_1 + r_2 + r_3 + \{f_2\}$$

which are mapped to lines of the ruling of the cubic scroll.

The directrix line of the scroll is the image of the (-2) -section e_2 , the blow up at q , which is mapped to L .

The family of quadrics containing these four lines through q forms a pencil, so through a general point in \mathbb{P}^3 there is only one of these quadrics. Since there are four possible choices of r_1 , r_2 and r_3 , there are four quadrics through $\tau(y)$ which are mapped to cubic scrolls in X . □

A description of case (E15) is given in the following proposition:

Proposition 2.3.7. *Let X be the Bronowski threefold corresponding to case (E15). Then X has a double line L , described by the vertices of cones of the one-dimensional family \mathcal{Q}' contained in X . There are three reducible quadrics in this family, each is a pair of planes.*

There are four smooth quadric surfaces in X that intersect the cones of \mathcal{Q}' in lines of one of its rulings.

The four double lines of \mathcal{X} in \mathbb{P}^3 are mapped to cubic scrolls S_3^x through x . A line of the ruling of each scroll lies on a quadric of \mathcal{Q}' and the directrix line is L .

Proof. Given four intersecting lines, there are six planes containing pairs of them. Then there are three reducible quadrics in \mathcal{Q} , that is, three pairs of planes. Together with Lemma 2.3.3, this proves the first part. The last part is the content of Lemma 2.3.5.

Let Π be a plane containing ℓ and a double line r of \mathcal{X} . There are four of such planes. Then:

$$\mathcal{X} \cap \Pi = l + 2r + \{\text{conics through } p \text{ and } q\}$$

and Π is mapped to a smooth quadric surface. A line in Π through p is mapped to a line intersecting the cones of \mathcal{Q}' . A line through q lies on a cone of \mathcal{Q} , so it is mapped to a line in a cone of \mathcal{Q}' . □

(E16) A double line and two simple lines

This case is now described:

Proposition 2.3.8. *Let X be the Bronowski threefold corresponding to case (E16). In this case, the one-dimensional family of quadric cones \mathcal{Q}' contained in X has two reducible quadrics, both are pairs of planes. One of them corresponds to the root with multiplicity two in the Segre symbol.*

Let R be the singular line of this quadric and let L be the line described by the vertices of cones of \mathcal{Q}' . Then R and L are double lines of X and they intersect each other in one point.

There are three smooth quadric surfaces in X that intersect the cones of \mathcal{Q}' in lines of one of its rulings.

One of the four double lines of \mathcal{X} in \mathbb{P}^3 is mapped to R . The other three lines are mapped to cubic scrolls S_3^x through x with directrix line L . A line of the ruling of each scroll lies on a quadric of \mathcal{Q}' . One of these scrolls contains R .

Proof. Let r be the double line. This means that there is a line r' infinitely near to r in the singular locus of \mathcal{X} . The special members of the pencil \mathcal{Q} are two pairs of planes. Each of these planes contains two lines of the singular locus of \mathcal{X} , so it is mapped to a plane in X . Let $Q \in \mathcal{Q}$ be the pair of planes which is singular in r . It corresponds to the root with multiplicity two in the Segre symbol. This follows from Proposition 2.1.6 applied to the associated pencil of conics in \mathbb{P}^2 .

The existence of the three smooth quadric surfaces in X follows, as before, from considering the planes through ℓ and a double line of \mathcal{X} . Since r' is infinitely near to r , there are only three of such planes.

The linear system \mathcal{X} intersects a general plane through r in $2r$ and cubics with a double point in q . Now blow up the point q , and then the line r . So far, the situation is pretty much the same as in the simple line case, and $E_r \cong \mathbb{F}_0$.

In E_r , the line $V_r = r'$, of type $(0, 1)$, is a fixed double line of \mathcal{X} . It follows then:

$$\mathcal{X}_r \equiv (1, 2) \equiv 2r' + (1, 0)$$

where $(1, 0)$ is the moving part, mapping E_r to a line R in X . It maps each point of r to a point in R . Note that $E_q \cap E_r$ is mapped to a point, so R intersects L in one point.

The line r' is a complete intersection and:

$$N_{r'} = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

In $E_{r'}$, both E_r and V intersect in lines of type $(0, 1)$. Hence $\mathcal{X}_{r'}$ is of type $(1, 2)$. It has a simple point in $E_q \cap E_{r'}$, lying on the curve $C = V_q$. So it is mapped to a cubic scroll S_3^x , like the one in the simple line case.

The reducible quadric Q does not contain the plane through r and r' . Then $Q_r \equiv (0, 2)$ does not contain V_r and Q is mapped to a pair of planes intersecting S_3^x in R .

It remains to prove that R is a double line of X .

Let q' be a general point of r and let Ω be a general plane through q' . As it was remarked above, a point of r is mapped to a point of R .

The intersection of \mathcal{X} with Ω consists of degree five curves with a double point in q' , a second double point q'' infinitely near to it, two other double points and two simple points. Blowing up q' and q'' , the curve $\Omega_{q'}$ has self intersection -2 in Ω and no intersection with $\mathcal{X} \cap \Omega$. Then Ω is mapped to a surface with a double point in $\sigma(q') \in R$. By Lemma 1.3.3, it is a double point of X . Hence R is a double line of X . □

(E17) Two double lines

All the information needed here is detailed in the previous section. Then we get:

Proposition 2.3.9. *Let X be the Bronowski threefold corresponding to case (E17). The one-dimensional family of quadric cones \mathcal{Q}' in X has a double plane and a pair of planes.*

Let L be the line described by the vertices of cones of \mathcal{Q}' , it is a double line of X . Two other lines, R and R' , which are contained in the double

plane of \mathcal{Q}' , are double lines of X . These two lines intersect in a point of L .

There are two smooth quadric surfaces in X that intersect the cones of \mathcal{Q}' in lines of one of its rulings.

Two of the four double lines of \mathcal{X} in \mathbb{P}^3 are mapped to R and R' . The other two lines, infinitely near to those, are mapped to cubic scrolls S_3^x through x . A line of the ruling of each scroll lies on a quadric of \mathcal{Q}' and the directrix line is L . One of these scrolls contains R , the other contains R' .

(E18) A triple and a simple line

Proposition 2.3.10. *Let X be the Bronowski threefold corresponding to case (E18). The only reducible quadric in \mathcal{Q}' is a pair of planes.*

Let R be the intersection of these two planes and let L be the line described by the vertices of cones of \mathcal{Q}' . Then X has multiplicity two in L , R and in a line R' infinitely near to R .

There are two smooth quadric surfaces in X that intersect the cones of \mathcal{Q}' in lines of one of its rulings.

Two infinitely near double lines of \mathcal{X} in \mathbb{P}^3 are mapped to R and R' . The other two lines are mapped to cubic scrolls S_3^x through x . A line of the ruling of each scroll lies on a quadric of \mathcal{Q}' and the directrix line is L . One of these scrolls contains R and R' .

Proof. Let r be the triple line. The situation is very similar to the previous cases. There is a line r' infinitely near to r and a third line r'' infinitely near to r' . These three lines are double lines of \mathcal{X} .

Blowing up q and r , the moving part of \mathcal{X}_r maps E_r to a line R . Blowing up r' , the moving part of $\mathcal{X}_{r'}$ maps $E_{r'}$ to a line R' . This line is infinitely near to R , since E_r and $E_{r'}$ intersect in a line of type $(1, 0)$, while the moving part of \mathcal{X} cut these surfaces in $(0, 1)$ -lines.

After the blow up at r'' the surface $E_{r''}$ is mapped to a cubic scroll S_3^x . It intersects the one pair of planes of \mathcal{Q}' in the line R .

To compute the multiplicity of X in R and R' , let Ω be a general plane through q' , a general point of r . Then $\mathcal{X} \cap \Omega$ has a double point in $q' = r \cap \Omega$, a second double point q'' infinitely near to it (corresponding to r') and a third double point infinitely near to q'' , corresponding to r'' . After the blow up at these three points, there are two intersecting curves with self-intersection -2 in Ω . These are mapped to two infinitely near double points in the image of Ω . By Lemma 1.3.3, these are double points of X . This proves that R and R' are double lines of X .

The quadric of \mathcal{Q}' containing R is the image of the quadric of \mathcal{Q} which is singular in r , namely, the pair of planes. The images of the planes spanned by ℓ and r and by ℓ and the other double line of \mathcal{X} are the two smooth quadric surfaces mentioned in the proposition. □

(E19) A line with multiplicity four

This case can be easily described with what was studied before. Therefore:

Proposition 2.3.11. *Let X be the Bronowski threefold corresponding to case (E19), in which the only reducible quadric in \mathcal{Q}' is a double plane.*

Let L be the line described by the vertices of cones of \mathcal{Q}' . Then L is a double line of X . There is another double line R in the double plane of \mathcal{Q}' a third double line R' infinitely near to it and a fourth double line R'' infinitely near to R' .

There is one smooth quadric surface in X that intersects the cones of \mathcal{Q}' in lines of one of its rulings.

Three double lines of \mathcal{X} in \mathbb{P}^3 are mapped to R , R' and R'' . The other line is mapped to a cubic scroll S_3^x through x . A line of its ruling lies on a quadric of \mathcal{Q}' and the directrix line is L . This scroll contains R , R' and R'' .

2.4 The classification

Let us now put together the varieties studied in this chapter.

Theorem 2.4.1. *Let X be a Bronowski threefold having as fundamental surface a quadric V . Suppose that the linear system \mathcal{X} defining the inverse of a general tangential projection of X has degree five.*

Then \mathcal{X} has multiplicity two in the base locus C_4 of a pencil of quadrics \mathcal{Q} and in a point $p \in V$, and multiplicity one in the two (possibly infinitely near) lines of V through p .

The threefold X has the following properties:

- (i) *It is an OADP variety;*
- (ii) *It is the residual intersection of the Segre Embedding of $\mathbb{P}^1 \times \mathbb{P}^3$ with a quadric of \mathbb{P}^7 containing a \mathbb{P}^3 of the ruling;*
- (iii) *The singularities of X have multiplicity two;*

- (iv) The pencil \mathcal{Q} is mapped to a one-dimensional family of quadric surfaces \mathcal{Q}' in X , which has general smooth member if and only if V is smooth;
- (v) If V is singular, then X has a double curve L which is described by the vertices of cones of \mathcal{Q}' . If the cones of \mathcal{Q} have the same vertex, this curve is a line; otherwise it is a conic and X is of type (E14);
- (vi) Apart from L , each component of the singular locus of X lies on the singular locus of a quadric of \mathcal{Q}' corresponding to a root of multiplicity greater than one in the Segre symbol.

In Table 2.1, the singularities of X are described, associated to each possible configuration of C_4 . Varieties (E1) to (E13) correspond to cases where V is smooth. In cases (E15) to (E19), the Segre symbol (with double brackets) refers to the pencil of conics cut by \mathcal{Q} on a general plane of \mathbb{P}^3 .

Proof. The Segre symbol is easily computable in each case, these computations can be found in [HP, p. 305]. Here, it is defined and explained in Section 2.1.4.

Besides that, Only items (i) and (ii) are in need of a proof, the other results were explained in the previous Sections.

In all the examples that were studied, the linear system \mathcal{X} is *relatively complete*, so X is linearly normal (cf. Remark 1.4.3). Then, by Theorem 1.4.4, X lies on a rational normal scroll Y , described by the spans of quadrics in \mathcal{Q}' . It is a four-dimensional scroll in \mathbb{P}^7 containing a one dimensional family of 3-spaces.

These 3-spaces are disjoint. Indeed, if they weren't, there would be two quadrics of \mathcal{Q}' spanning a \mathbb{P}^6 , none of them containing the general point x . This implies that there is a surface of \mathcal{X} containing two quadrics of \mathcal{Q} different from V . Such surface should then be the union of these two quadrics and the plane containing ℓ and ℓ' . However, this surface has multiplicity one in p , so it cannot belong to \mathcal{X} , a contradiction.

Hence Y is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$. Item (ii) now follows from the fact that X has degree seven and is a divisor on Y that intersects a general \mathbb{P}^3 of the ruling in a quadric.

Now, (i) follows from (ii), using the same argument of [CMR, Proposition 2.5].

□

String	C_4	Segre symbol	Singularities of X
(E1)	smooth quartic	$[1, 1, 1, 1]$	none
(E2)	nodal quartic	$[2, 1, 1]$	one point
(E3)	cuspidal quartic	$[3, 1]$	one point
(E4)	twisted cubic and a transversal line	$[2, 2]$	two points
(E5)	twisted cubic and a tangent line	$[4]$	two infinitely near points
(E6)	two transversal conics	$[(11), 1, 1]$	two points
(E7)	two tangent conics	$[(21), 1]$	two infinitely near points
(E8)	one conic and two lines intersecting it in two points	$[(11), 2]$	three points
(E9)	one conic and two lines intersecting it in one point	$[(31)]$	one point and three infinitely near to it
(E10)	four lines	$[(11), (11)]$	four points
(E11)	one double and two simple lines	$[(22)]$	one line
(E12)	two double lines	$[(211)]$	two lines
(E13)	one double conic	$[(111), 1]$	one conic
(E14)	a double line and a conic	–	one conic
(E15)	four lines	$[[1, 1, 1]]$	one line
(E16)	one double and two simple lines	$[[2, 1]]$	two lines
(E17)	two double lines	$[[(11), 1]]$	three lines
(E18)	one triple and one simple lines	$[[3]]$	three lines: L , $R \prec R'$
(E19)	one line with multiplicity four	$[[(21)]]$	four lines: L , $R \prec R' \prec R''$

Table 2.1: Varieties satisfying hypothesis (H) and having quadratic fundamental surface.

Chapter 3

The cubic case

In this chapter we consider the case in which the fundamental surface V is a cubic. This is the situation of the degree eight scroll in lines, described in Example 1.4.2. According to hypothesis (H), \mathcal{X} has degree seven. We'll also suppose that the base locus of \mathcal{X} has pure dimension one (see Proposition 3.1.3).

3.1 Base locus of \mathcal{X}

Since V is a cubic surface, the map $\bar{\tau} : E \cong \mathbb{P}^2 \dashrightarrow V \subset \mathbb{P}^3$ is defined by a linear system of conics with one base point. It is not complete, since the complete linear system maps to the scroll $S(1, 2) \subset \mathbb{P}^4$. Then V is a non normal cubic surface, a projection of $S(1, 2)$ from an external point.

The proof of the following proposition can be found in [Do1]:

Proposition 3.1.1. *Let $V \subset \mathbb{P}^3$ be a projection of the cubic scroll $S(1, 2)$ from an external point and let $r \subset V$ be the projection of the directrix line of the scroll. Then V has a double line s , projection of a conic of $S(1, 2)$ and it is projectively equivalent to one of the following surfaces:*

- (i) $x_0^2 x_2 + x_1^2 x_3 = 0$
- (ii) $x_0^2 x_3 + x_0 x_1 x_2 + x_1^3 = 0$

Moreover, in (i) (the general case), r and s are skew lines. In (ii), the conic of $S(1, 2)$ that is projected to the double line s is the union of the directrix line and a line of the ruling of the scroll, so r is infinitely near to s . The surface (ii) is called Cayley's ruled cubic.

When computing self-intersections of curves in V , we'll consider its normalization $S(1, 2)$.

Let p be the base point of $\Pi_{X,x}$. Then $\bar{\tau}$ maps lines through p to lines of the ruling of V . The point p is blown up and mapped to r . The image of a general line is a conic. But one of these lines, name it \hat{s} , is mapped to the double line s .

If V is a general projection of $S(1, 2)$, \hat{s} does not contain p . Then the map:

$$\bar{\tau}|_{\hat{s}} : \hat{s} \rightarrow s$$

gives a double cover of \mathbb{P}^1 , ramified over two points q_1^s and q_2^s in s .

If V is the Cayley's ruled cubic, then \hat{s} is a line through p . After the blow up at p , the union of \hat{s} and the exceptional divisor is mapped to s .

Looking at plane sections of V , we have:

Lemma 3.1.2. *A conic in E through p lies on the linear system $\Pi_{X,x}$ if and only if its two points of intersection with \hat{s} are mapped by $\bar{\tau}$ to the same point of s .*

In the Cayley's ruled cubic case, since p lies on \hat{s} , these two points consist of a proper point in \hat{s} and a point infinitely near to p .

Proof. A conic of $\Pi_{X,x}$ is mapped to a plane section of V , which intersects s in a double point. Then the two points of intersection of such conic with \hat{s} are mapped to the same point of s .

Conversely, a conic of E through p is mapped to a cubic curve. If it intersects \hat{s} in two corresponding points, the cubic curve will have a double point in s . Hence it is a plane cubic, and therefore a plane section of V .

In the Cayley's ruled cubic case, a conic through p intersects \hat{s} in p and in another point. Since p is blown up, the points infinitely near to p are mapped to distinct points, giving the last assertion. □

Let \tilde{E} be the plane E blown up at p , and write (a, b) for the class of a curve in \tilde{E} of degree a intersecting the exceptional divisor in b points. So a curve of type $(1, 1)$ is mapped by $\bar{\tau}$ to a line of the ruling of V , a curve of type $(1, 0)$ is mapped to a conic (or to $2s$), a curve of type $(2, 1)$ is mapped to a cubic (a plane section of V if it lies on $\Pi_{X,x}$) and the exceptional curve $(0, -1)$ is mapped to r . Abusing notation, the same terminology will be used for curves in E and in V .

Note that also in case (ii) the line \hat{s} in \tilde{E} is of type $(1, 0)$. It is the union of a line through p and the exceptional line, so $(1, 0) = (1, 1) + (0, -1)$.

Below is a description of the linear system \mathcal{K} :

Proposition 3.1.3. *Let s be the double line of V and let r be the image of the blow up at the base point of $\Pi_{X,x}$. Suppose the base locus of \mathcal{X} has pure dimension 1. Then \mathcal{X} is the linear system of surfaces with degree seven having:*

- *multiplicity four in s*
- *multiplicity two in C_6*
- *multiplicity one in r ,*

where $C_6 \subset V$ is a curve of type $(5, 4)$, that is, it is the image of a quintic in E having multiplicity four in p . In particular, C_6 cuts s in five points and r in four points, supposing it does not contain these lines.

Proof. Following the notation of Lemma 1.3.2, the linear system \mathcal{X}' must desingularize the cubic V , so $\text{mult}_s \mathcal{X}' \geq 1$. This implies that $\text{mult}_s \mathcal{X} \geq 3$. On the other hand, the multiplicity of \mathcal{X} in s is at most 4, since hypothesis (H) implies that \mathcal{X}'' has no base locus. Therefore:

$$m = \text{mult}_s \mathcal{X} \in \{3, 4\}$$

As remarked in Lemma 1.3.2, the moving part of $\mathcal{X}' \cap V$ defines the inverse of $\bar{\tau}$. Then there is a curve $C \subset V$ such that:

$$\mathcal{X}' \cap V = 2(m - 2)s + \{\text{conics}\} + C$$

In particular, C is a double curve of \mathcal{X} .

If $m = 3$, then C has degree eight and \mathcal{X} intersects V in a curve of degree at least $2 \cdot 3 + 2 \cdot 8 = 22$, which is not possible. Therefore, $m = 4$, $C = C_6$ has degree six and:

$$\mathcal{X}' \cap V = 4s + \{\text{conics}\} + C_6$$

The intersection of V with a hypersurface of degree d has class $(2d, d)$. Then, for the intersection of \mathcal{X}' with V , we have:

$$(8, 4) \equiv \mathcal{X}' \cap V = 4s + \{\text{conics}\} + C_6 \equiv (2, 0) + (1, 0) + (a, b)$$

which gives $C_6 \equiv (5, 4)$.

The intersection with \mathcal{X} must be a fixed divisor of V , then:

$$(14, 7) \equiv \mathcal{X} \cap V = 8s + 2C_6 + \{\text{fixed line}\} \equiv (4, 0) + (10, 8) + (a, b)$$

so the fixed line has type $(0, -1)$, that is, it is r . This finishes the proof, since hypothesis (H) implies that the base locus of \mathcal{X} lies on V . □

Note however that C_6 can contain r or s as a component, as is now described:

Lemma 3.1.4. *If $r \subset C_6$, then there is another line r' infinitely near to r in the base locus of \mathcal{X} and C_6 consists of r and five lines of the ruling. In particular:*

$$\mathcal{X} \cap V = 2 \sum_{i=1}^5 \ell_i + 8s + 3r$$

and C_6 cuts s in five points.

If $s \subset C_6$, then there is another line s' infinitely near to s , which is a double line of \mathcal{X} . In this case, C_6 is the union of s with multiplicity two and four lines of the ruling and:

$$\mathcal{X} \cap V = 2 \sum_{i=1}^4 \ell_i + 12s + r$$

In particular, C_6 cuts r in four points.

Proof. Suppose first that $r \subset C_6$. Since $r \equiv (0, -1)$, we have that:

$$C_6 \equiv (5, 4) \equiv (0, -1) + (5, 5)$$

So C_6 is the union of r and five lines ℓ_1, \dots, ℓ_5 of the ruling.

Note that \mathcal{X} has multiplicity two in r . Blowing up r , we have that $\mathcal{X}_r \equiv (5, 2)$ and $V_r \equiv (2, 1)$. But these two curves intersect in five double points, corresponding to the five lines in C_6 . Since $(5, 2) \cdot (2, 1) = 9$ and V_r is irreducible, it follows that \mathcal{X}_r contains $V_r = r'$.

Suppose now that $s \subset C_6$. But s is a double line of V , so we must have $2s \subset C_6$. Since $2s \equiv (1, 0)$, then:

$$C_6 \equiv (5, 4) \equiv (1, 0) + (4, 4)$$

and C_6 is the union of $2s$ and four lines ℓ_1, \dots, ℓ_4 of the ruling. Since \mathcal{X} has multiplicity four in s and two in C_6 , it follows that:

$$\mathcal{X} \cap V = 2 \sum_{i=1}^4 \ell_i + 12s + r$$

So blowing up s , $V_s \equiv (1, 2)$ is irreducible and $\mathcal{X}_s \equiv (3, 4)$ contains $2V_s = 2s'$, that is:

$$\mathcal{X}_s \equiv 2s' + \{(1, 0)\}$$

and the result follows. □

We can now give a simple description of the possibilities for the base locus of \mathcal{X} :

Corollary 3.1.5. *Let C_6 be the sextic curve in the base locus of \mathcal{X} , given by Proposition 3.1.3. Then one of the following holds:*

- (i) C_6 is the union of r and five lines of the ruling;
- (ii) C_6 is the union of $2s$ and four lines of the ruling;
- (iii) C_6 is the union of an irreducible curve of degree d , which is smooth outside s , and $6 - d$ lines of the ruling, for $2 \leq d \leq 6$.

Proof. Items (i) and (ii) have been proved in Lemma 3.1.4.

Consider the curve $\bar{\sigma}(C_6)$ in $E \cong \mathbb{P}^2$, having degree five and multiplicity four in p , that is, a $(5, 4)$ curve. If it contains an irreducible curve \hat{C} of degree d' , this curve must have multiplicity $d' - 1$ in p and no other singularities, since:

$$(5, 4) - (d', d' - 1) = (5 - d', 5 - d')$$

So $\bar{\sigma}(C_6)$ is the union of \hat{C} and $5 - d'$ lines through p . Then the result follows with $d = d' + 1$. □

Another easy consequence of Lemma 3.1.3 and Lemma 3.1.4 is the following:

Corollary 3.1.6. *If V is a cubic surface, then the associated threefold X is ruled by lines.*

Proof. Let y be a general point of X and set $y' = \tau(y)$. Consider the plane Π spanned by s and y' .

If C_6 does not contain s , it intersects Π in five points in s and a sixth point p . Let ℓ be the line spanned by p and y' . Then \mathcal{X} intersects ℓ in p with multiplicity two, in $\ell \cap s$ with multiplicity four and in a moving point. Therefore ℓ is mapped to a line through y .

Suppose now that C_6 contains s and consider the blow up at s . Then $\Pi_s \equiv (0, 1)$ intersects $s' = V_s \equiv (1, 2)$ in one point, lying on the fiber over a point $p \in s$. Then the line through p and y' is intersected by \mathcal{X} in p with multiplicity four and a double point infinitely near to it. Therefore it is mapped to a line in X through y . □

3.2 The general case

Throughout this section, suppose V is of type (i) in Proposition 3.1.1, so r and s are skew lines. We will first study specific properties of V and the images of r and s in X .

Next, we identify the possible singularities of X based on properties of the base locus of \mathcal{X} and present a family of surfaces contained in X which helps to understand its geometry. Finally, a specific example will be studied in detail.

3.2.1 Some properties

Let's first give more details on V :

Lemma 3.2.1. *The surface V is ruled by lines that intersect both s and r . There is only one line of the ruling through each of the points q_1^s and q_2^s of s . Through any other point of s there are two lines of the ruling. Apart from these pairs, the lines of the ruling are disjoint.*

Proof. The lines of the ruling are images of lines in E through p . Since p does not lie on \hat{s} , these cut \hat{s} in a point different from p .

Given a general point in s , it has two distinct preimages in \hat{s} . If the point is q_1^s or q_2^s , it has only one preimage in \hat{s} . Then the result follows. \square

A plane section of V is a cubic with a singular point in s . More precisely:

Lemma 3.2.2. *Let q be a point in s . If $q \neq q_1^s, q_2^s$, then the tangent cone of V in q is a pair of planes, each containing s and one of the lines of the ruling through q . Moreover, a general plane section of V through q is a nodal cubic.*

If q is q_1^s or q_2^s , the tangent cone of V in q is a double plane, the plane spanned by s and the line of the ruling through q . A general plane section of V through q is a cuspidal cubic.

Proof. In both situations, s is a double line of the tangent cone of V in q .

In the first case, the tangent cone must contain both lines of the ruling through q , so it is the pair of planes containing s and one of these lines. A general plane through q cuts these planes in two distinct lines, which lie on the tangent cone of the cubic curve in q . Then it is a nodal point.

Suppose now that q is q_1^s or q_2^s . A plane containing s cuts V in $2s$ and a line of the ruling. If such plane lies on the tangent cone of q , this intersection

has multiplicity three in q , so the unique line of the ruling through q is contained in this plane. Then the tangent cone is the claimed double plane, name it Π .

A general plane section through q is a cubic with tangent cone at q contained in Π . But the cubic itself cannot lie on Π , so the double point is a cusp. □

The next step is to determine the images via σ of the lines r and s . By Proposition 3.1.3, if C_6 does not contain these lines, it cuts r in four points and s in five points. Remember that the case in which one of these is contained in C_6 is described in Lemma 3.1.4.

Proposition 3.2.3. *If r is not contained in C_6 , it is mapped by σ to the line ℓ_x through x . This line corresponds to the base point of $\Pi_{X,x}$, which was mapped to r in the first place. If $r \subset C_6$ it is mapped to a plane and the line r' infinitely near to r is mapped to ℓ_x . These two intersect in a point.*

If s is not contained in C_6 , it is mapped to a weak Del Pezzo surface of degree four D_4^x through x . If $s \subset C_6$, then it is mapped to a line L . The line s' infinitely near to s is mapped to a quartic surface D_4^x , a projection of the Veronese quartic surface having multiplicity two along L .

Since r and s are disjoint, ℓ_x and D_4^x intersect only in x .

Proof. Remember that the base locus of \mathcal{X} has no embedded points. Suppose first that r is not contained in C_6 .

Blowing up r , \mathcal{X} cuts $E_r \cong \mathbb{F}_0$ in curves of type $(6, 1)$ having four double points at the intersections with C_6 . So they break in the fibres over these points plus $(2, 1)$ curves with four base points.

Applying Lemma 1.1.1, the moving part corresponds, in \mathbb{P}^2 , to cubics with five simple points and one double point. After two standard quadratic maps, we get a linear system of lines through a point. So E_r is mapped to a line. The cubic V cuts the exceptional divisor in a $(2, 1)$ -curve through the four base points, so it is contracted to x .

The fixed components intersect the moving part in the base points, so they do not affect the image of E_r .

Now suppose $r \subset C_6$. By Lemma 3.1.4, C_6 is the union of r and five lines of the ruling.

Blow up r . As already noted:

$$\mathcal{X}_r \equiv (5, 2) \equiv r' + (3, 1)$$

and the moving part has five base points in the intersections with the five lines. It maps E_r to a plane.

To find the image of r' , first blow up the five lines ℓ_i . Now, r' is the complete intersection of E_r blown up at five points with V . In E_r , $(r')^2 = 4$, so after the blow ups we have $(r')^2 = -1$. In V , $(r')^2 = -1$. Hence, blowing up r' gives $E_{r'} \cong \mathbb{F}_0$.

In E_r , \mathcal{X} intersects r' in its five base points. So after the blow ups there is no intersection, and $\mathcal{X}_{r'} \equiv (0, 1)$. Hence r' is mapped to a line. This is the one line ℓ_x through x , since $V_{r'} \equiv (0, 1)$ is contracted to x . Moreover, $(E_r)_{r'} \equiv (0, 1)$, so the line intersects the plane in one point.

This proves the first part.

For the second part, suppose first that s is not contained in C_6 .

Consider the blow up at s . \mathcal{X}_s is made of curves of bidegree $(3, 4)$, having five double points at the intersections with C_6 . Depending on C_6 , some of these points can be infinitely near. The linear system is birationally equivalent to cubics in \mathbb{P}^2 with five base points. V intersects the exceptional surface in a $(1, 2)$ curve through the five points, which is contracted to x . Then s is mapped to a weak Del Pezzo quartic surface through x .

Now suppose $s \subset C_6$. By Lemma 3.1.4, C_6 is the union of $2s$ and four lines of the ruling. Blowing up s , we have:

$$\mathcal{X}_s \equiv 2s' + \{(1, 0)\} \equiv (3, 4)$$

where $s' = V_s \equiv (1, 2)$. The moving lines of type $(1, 0)$ cut the rational curve s' in a linear series of degree two and projective dimension one. This linear series is not complete, otherwise it would have dimension two.

In E_s , $(s')^2 = 4$. In V , $(s')^2 = 1$, since the self-intersection of a conic in $S(1, 2)$ is 1. So blowing up $s' = V \cap E_s$ gives $E_{s'} \cong \mathbb{F}_3$. $\mathcal{X}_{s'}$ cuts $(E_s)_{s'} \equiv e_3$ in two points, cuts a fiber f_3 in two points and cuts $V_{s'} \equiv e_3 + 3f_3$ in four fixed double points. These double points are the intersections of the four double lines of \mathcal{X} with $E_{s'}$. Then:

$$\mathcal{X}_{s'} \equiv 2e_3 + 8f_3$$

having four double base points.

Using Lemma 1.1.1, $\mathcal{X}_{s'}$ corresponds, in \mathbb{P}^2 , to degree eight curves with four double points, one point of multiplicity six and two double points infinitely near to it. After three standard quadratic transformations, one maps these curves to conics with no base points.

As noted above, before the blow up at s' , the moving part of \mathcal{X}_s intersected it in a non complete linear series. Then, after this blow up, the

restriction of $\mathcal{X}_{s'}$ to $(E_s)_{s'} \equiv e_3$ is a non complete linear series of degree two and dimension one, so it maps the (-3) section to a double line, instead of a conic. Therefore, the base point free linear system of conics we have found is not complete and $\mathcal{X}_{s'}$ maps $E_{s'}$ to a projection of a Veronese surface with a double line in the image of e_3 . □

3.2.2 Singularities of X

Note that by hypothesis (H) X is normal, therefore regular in codimension one. Remember that we are supposing that the base locus of \mathcal{X} has no embedded points. The following result will guide our search for singularities of X .

Lemma 3.2.4. *Let x_q be an isolated singular point or a general point in a singular curve of X . Then x_q does not lie on the base locus of τ and it is mapped to a singular point q of C_6 not lying on r . This includes general points in non reduced components of C_6 .*

Proof. If x_q is an isolated singular point of X , by Terracini's Lemma it cannot lie on a general tangent space of X , therefore it does not lie on the base locus of τ . For the same reason, a singular curve cannot lie on a general tangent space, so a general point of it does not lie on the base locus of τ . It may happen though that $T_x X$ contains one or more points of this curve, but it does not contain general points.

Since σ (the inverse of τ) restricted to $\mathbb{P}^3 \setminus V$ is an isomorphism, $q = \tau(x_q)$ is contained in V . Since $\sigma(V) = x$ and X is smooth in x , q lies on the base locus of \mathcal{X} . Remember that this base locus has no isolated points. In other words, x_q lies on a surface in X that is contracted by τ to a curve in V .

The line r is mapped by σ to the line ℓ_x through x , which lies on $T_x X$. Therefore $q \notin r$.

If q is not a singular point of C_6 , we will prove that x_q is a smooth point of a general tangent hyperplane section of X at x through x_q . In particular, x_q is not a singular point of X .

Since x_q is either an isolated singular point or a general point of a singular curve of X , a general tangent hyperplane section of X at x through x_q is mapped by τ to a general plane through q . So, in order to prove the assertion, we study the image of a general plane Ω through q .

Note that a general plane is cut by \mathcal{X} in degree seven curves having six double points, one point of multiplicity four and one simple base point. This

is the case of Ω if q is a smooth point of C_6 not lying on s or a point in $s \setminus C_6$, so this plane is mapped to a smooth hyperplane section of X .

Suppose then that q is a smooth point of C_6 lying on s (in particular, $s \not\subset C_6$). Then $\mathcal{X} \cap \Omega$ consists of degree seven curves with multiplicity four in $q = s \cap \Omega$, six double points in $C_6 \cap \Omega$ and one simple point in $r \cap \Omega$. Since $q \in C_6$, one of the six double points q' lies infinitely near to q . This can also be verified by blowing up q and noting that \mathcal{X}_q consists of the line through s_q and $(C_6)_q$ with multiplicity two and two moving lines through s_q (see the proof of Lemma 3.2.6). Since Ω_q is a general line of E_q , \mathcal{X}_q intersects it in a fixed double point and two moving points. This explains the double point q' of $\mathcal{X} \cap \Omega$ infinitely near to q . Since q is a smooth point of C_6 , $\mathcal{X} \cap \Omega$ has no other double points infinitely near to q or q' .

Now it easily follows that the image of Ω is not singular in x_q . Blowing up q , the exceptional curve $\Omega_q \subset \Omega$ is mapped to a conic. Blowing up q' , this second exceptional curve is also mapped to a conic. Then none of these curves are contracted. Moreover, σ is an isomorphism outside V , so the only curve in Ω through q that is contracted is $V \cap \Omega$, which is mapped to the smooth point x . Hence x_q is a smooth point in the image of Ω . \square

We will begin studying the singularities of C_6 that do not lie on s :

Proposition 3.2.5. *Let $q \in C_6$ be a singular point of C_6 outside s . Then q is mapped to a double point x_q of X . Moreover, if q does not lie on a non reduced component of C_6 , then the tangent cone of X in x_q has rank four.*

Proof. By Corollary 3.1.5, C_6 is either: the union of r and five lines of the ruling of V ; the union of $2s$ and four lines of the ruling; or the union of a curve of degree d , which is smooth outside s , and $6 - d$ lines of the ruling. As seen in the proof of this result, this curve is of type $(d - 1, d - 2)$, so it intersects a line of the ruling in one point (possibly in a second point lying on s , since $\bar{\tau}$ is $2 : 1$ in s). Therefore each intersection outside s of this degree d curve with a line of the ruling is transversal.

Therefore, if q is a singular point of C_6 not lying on s , it is either:

- (a) the transversal intersection of a line of the ruling of V and an irreducible curve of degree $d > 1$, which is smooth outside s ;
- (b) the transversal intersection of a line of the ruling of V and $r \subset C_6$;
- (c) a point in a line of the ruling of V which is a non reduced component of C_6 (and possibly (a) or (b)).

Remember that, by Proposition 3.1.3, \mathcal{X} has multiplicity two in C_6 . Then in all three cases above, blowing up q , \mathcal{X}_q has degree two and has two or more double points. These points lie on the line V_q , since V is smooth outside s . Then $\mathcal{X}_q = 2V_q$ and q is mapped to a point $x_q \in X$.

To prove that x_q is a double point, we'll use Lemma 1.3.3. Let Ω be a general plane containing q . The curves $\mathcal{X} \cap \Omega$ have multiplicity two in q . Blowing up q , there is another double point in $e = E_q \cap \Omega$, since $\mathcal{X}_q = 2V_q$. Blowing up this point, we get $e^2 = -2$ in Ω , with $\mathcal{X} \cap \Omega$ having no intersection with e . So e is mapped to a double point in the image of Ω . There may be other infinitely near double points. By Lemma 1.3.3, x_q is a double point of X .

Now we have to prove that, in cases (a) and (b), the tangent cone of X in x_q is a rank four quadric cone. This will be done using Lemma 1.1.4.

Let ℓ be the line of the ruling through q and let C be the other component of C_6 through q . In case (b), $C = r$ and \mathcal{X} contains a line r' infinitely near to r (see Proposition 3.2.3). Let $\widehat{\mathcal{X}}$ be the linear system of surfaces in \mathcal{X} having multiplicity three in q .

Blowing up q , in both cases $\widehat{\mathcal{X}}_q$ has multiplicity two in ℓ_q and C_q . In case (b) it has a further base point infinitely near to C_q , corresponding to r' . Note that C_q and ℓ_q are not infinitely near, since the intersection of C and ℓ is transversal. All base points must lie on $t = V_q$, which is a line, since $q \notin s$. Then $\widehat{\mathcal{X}}_q$ must contain t and:

$$\widehat{\mathcal{X}}_q = t + \{\text{conics through } \ell_q \text{ and } C_q\}$$

The moving conics map E_q to a smooth quadric surface.

Since we know that x_q is a double point of X , Lemma 1.1.4 implies that the projectivization of the tangent cone of X in x_q is a smooth quadric surface. Therefore the rank of the tangent cone is four. □

Next, we discuss the case in which q lies on $C_6 \cap s$. We will first study the intersection of a component of C_6 with the lines of V through a smooth point of this component.

Lemma 3.2.6. *Let ℓ and ℓ' be distinct lines of the ruling of V intersecting in a point $q \in s$. Let C be an irreducible component of C_6 having degree $d > 1$ and being smooth in q . Then, besides q , C intersects only one of these lines, say ℓ , in another point.*

Moreover, after the blow up at q , E_q is mapped to a point in X if and only if $\ell \subset C_6$. If this is not the case, \mathcal{X}_q has a fixed double line t' . On the

other hand, if $\ell' \subset C_6$ then, blowing up s and t' , $E_{t'}$ is mapped to a point in X .

Proof. As seen in the proof of Corollary 3.1.5, C is the image via $\bar{\tau}$ of a curve \hat{C} in $E \cong \mathbb{P}^2$ of degree $d - 1$ having multiplicity $d - 2$ in p . The two lines ℓ and ℓ' are images of lines $\hat{\ell}$ and $\hat{\ell}'$ through p .

Let $\hat{q} = \hat{\ell} \cap \hat{s}$ and $\hat{q}' = \hat{\ell}' \cap \hat{s}$, which are different points, since ℓ and ℓ' are distinct. Then both \hat{q} and \hat{q}' are mapped by $\bar{\tau}$ to $q \in s$. Since C is smooth in q , \hat{C} passes through only one of these points, say \hat{q}' . Therefore:

$$\begin{aligned}\hat{C} \cap \hat{\ell} &= (d - 2)p + \{\text{one point}\} \\ \hat{C} \cap \hat{\ell}' &= (d - 2)p + \hat{q}'\end{aligned}$$

Then it follows that C intersects ℓ in q and in a second point and intersects ℓ' only in q . This proves the first part.

By Lemma 3.2.2, the tangent cone of V in q is the union of the plane Π containing s and ℓ , and the plane Π' containing s and ℓ' . In particular, the tangent line $T_q C$ of C in q lies on one of these planes.

Since C is of type $(d - 1, d - 2)$, it intersects s in $d - 1$ points, including q . Note that $T_q C$ does not lie on $\Pi \setminus s$. Indeed, this would imply that:

$$C \cap \Pi = 2q + \{d - 2 \text{ points}\} + q'$$

where q' is the second point of intersection of C with ℓ . Then C would be contained in Π . But $V \cap \Pi = 2s + \ell$. Hence $T_q C \in \Pi'$.

Blow up q . Set $t = \Pi_q$ and $t' = \Pi'_q$. Then \mathcal{X}_q has multiplicity four in $s_q = t \cap t'$ and multiplicity two in $C_q \in t'$.

If $l \subset C_6$, then \mathcal{X}_q has also a double point in $\ell_q \in t$, which implies that:

$$\mathcal{X}_q = 2t + 2t'$$

and E_q is mapped to a point.

On the other hand, if ℓ is not contained in C_6 the only other possible components through q are ℓ' and lines infinitely near to it (see Corollary 3.1.5). So \mathcal{X}_q will only have further double points in t' and:

$$\mathcal{X}_q = 2t' + \{\text{pairs of lines through } s_q\}$$

which maps E_q to a conic.

Now suppose $\ell' \subset C_6$. Blow up s , which has multiplicity four for \mathcal{X} . Now, the line $t' = \Pi' \cap E_q$ has normal bundle:

$$N_{t'} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

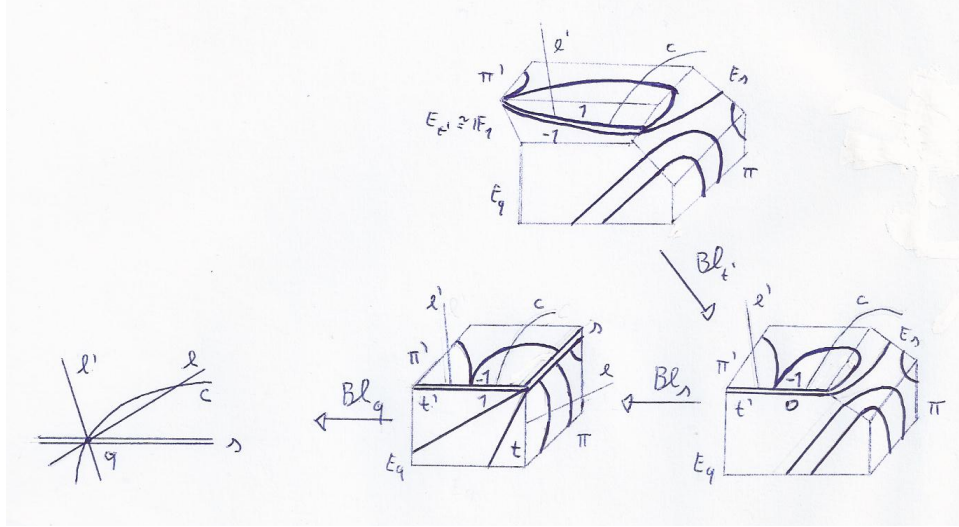


Figure 3.1: Blow up of \mathcal{X} in $q \in C \cap s$ when $\ell \not\subset C_6$ and $\ell' \subset C_6$

Blow up t' . In $E_{t'} \cong \mathbb{F}_1$, $\mathcal{X}_{t'}$ has no intersection with $(E_q)_{t'} \equiv e_1$, then $\mathcal{X}_{t'} \equiv 2e_1 + 2f_1$. It has double points in the intersections with ℓ' and C , so it is a fixed double curve and $E_{t'}$ is mapped to a point.

Figure 3.1 illustrates these blow ups when $\ell \not\subset C_6$, but $\ell' \subset C_6$. □

Before studying the next type of singularity, we give a Lemma and make some considerations:

Lemma 3.2.7. *Let C be a non-degenerate reduced irreducible curve of degree six in \mathbb{P}^3 . Suppose that either C is rational and has a 5-secant line or that $p_a(C) \geq 1$. Then there are at least two distinct cubic surfaces containing C .*

Proof. The exact sequence:

$$0 \rightarrow \mathcal{I}_{C/\mathbb{P}^3}(3) \rightarrow \mathcal{O}_{\mathbb{P}^3}(3) \rightarrow \mathcal{O}_C(3) \rightarrow 0$$

gives the cohomology exact sequence:

$$0 \rightarrow H^0(\mathcal{I}_{C/\mathbb{P}^3}(3)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \rightarrow H^0(\mathcal{O}_C(3)) \rightarrow H^1(\mathcal{I}_{C/\mathbb{P}^3}(3)) \rightarrow 0$$

By Riemann-Roch, we have:

$$h^0(\mathcal{O}_C(3)) = h^1(\mathcal{O}_C(3)) + 18 + 1 - p_a(C) = 19 - p_a(C)$$

since $2p_a(C) - 2 \leq 6 < \deg(\mathcal{O}_C(3))$ gives $h^1(\mathcal{O}_C(3)) = 0$.

On the other hand, $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = \binom{6}{3} = 20$. If $p_a(C) \geq 1$, then:

$$h^0(\mathcal{I}_{C/\mathbb{P}^3}) = 20 - 19 + p_a(C) + h^1(\mathcal{I}_{C/\mathbb{P}^3}) \geq 2$$

If C is rational, then by [GLP] $h^1(\mathcal{I}_{C/\mathbb{P}^3}) > 0$ if and only if it has a 5-secant line, giving, in this case:

$$h^0(\mathcal{I}_{C/\mathbb{P}^3}) = 20 - 19 + 0 + h^1(\mathcal{I}_{C/\mathbb{P}^3}) > 1$$

Hence in both cases there are at least two different cubic surfaces containing C . □

If we are in the general case, that is, if C_6 is smooth, then it is rational and has a 5-secant line, namely s . Then this Lemma shows that there is a cubic surface other than V containing C_6 . Therefore there is such a cubic surface in the other special cases (e.g. C_6 reducible). We will not give the details here.

Note that any cubic surface containing C_6 must also contain the lines r and s . Indeed, if none of these lines is contained in C_6 , then C_6 intersects them in five and four points respectively. If $s \subset C_6$, then C_6 contains four lines of the ruling of V , intersecting r in four points. And if $r \subset C_6$, there are five lines of the ruling in C_6 , so it intersects s in five points.

This implies that, if S is a cubic surface different from V containing C_6 , then:

$$S \cap V = C_6 + 2s + r \tag{3.1}$$

where the multiplicity two in s follows from the fact that V is singular along s . Note that the general element of the pencil of cubic surfaces generated by V and S is irreducible, since V is.

Next, the case in which a reduced component C of C_6 is singular in q will be studied. As shown in the proof of Corollary 3.1.5, C is of type $(d-1, d-2)$, so its image in $E \cong \mathbb{P}^2$ via $\bar{\sigma}$ has no singular points, except for p . Then C is singular in q if only if $q \in s$ and $\bar{\sigma}(C)$ either contains the two points of \hat{s} that are mapped to $q \notin \{q_1^s, q_2^s\}$ or is tangent to \hat{s} in $\bar{\sigma}(q)$, with $q \in \{q_1^s, q_2^s\}$. In any of these cases, q is a double point of C , since the map $\bar{\tau}$ restricted to \hat{s} is $2:1$.

Lemma 3.2.8. *Let C be an irreducible component of C_6 with degree $d > 1$ and q be a point in $C \cap s$.*

Suppose q is different from q_1^s or q_2^s . Let Π_1 and Π_2 be the two planes of the tangent cone of V in q . After blowing up q , set $t_i = \Pi_i \cap E_q$, for $i = 1, 2$. If C is singular in q , then one of the following holds:

- (i) the tangent cone of C in q is a pair of lines different from s , one lying on Π_1 and the other in Π_2 ;
- (ii) the tangent cone of C in q is s and another line lying on one of the planes, say Π_1 . Then after blowing up q and s , C intersects $E_s \cap E_q$ in $(t_2)_s$;
- (iii) the tangent cone of C in q is $2s$. After blowing up q and s , C intersects $E_s \cap E_q$ in $(t_1)_s$ and $(t_2)_s$.

On the other hand, if q is q_1^s or q_2^s and C is singular in q , the tangent cone of C in q is a double line; if C is smooth in q , it is tangent to the line of the ruling of V through this point.

Proof. Let \hat{C} be the strict transform of C in E via $\bar{\tau}$, a curve of degree $d - 1 \leq 5$ having multiplicity $d - 2$ in p (see the proof of Corollary 3.1.5).

Suppose q is not q_1^s or q_2^s . Then it has two preimages via $\bar{\tau}$, say $\hat{q}_1, \hat{q}_2 \in \hat{s}$. Let $\ell_1 \in \Pi_1$ and $\ell_2 \in \Pi_2$ be the two lines of the ruling through q , such that the strict transform of ℓ_i via $\bar{\tau}$ passes through \hat{q}_i .

We will first give a correspondence between lines through \hat{q}_1 in E and lines through q in Π_1 . If the line in Π_1 is s or ℓ_1 , it corresponds to its strict transform in E . Given a line ℓ through q in Π_1 different from s or ℓ_1 , ℓ is the tangent line at q of a smooth conic C_ℓ . Then ℓ corresponds to the strict transform of C_ℓ via $\bar{\tau}$, which is a line $\hat{\ell}$ through \hat{q}_1 .

Now, if a curve D is tangent to $\hat{\ell}$ in \hat{q}_1 , then $\bar{\tau}(D)$ is tangent to C_ℓ in q . In other words, this correspondence associates to a tangent line of a curve in \hat{q}_1 a line in the tangent cone at q of the image of this curve via $\bar{\tau}$.

The same reasoning gives a correspondence of lines through q in Π_2 and lines through \hat{q}_2 in E .

Since \hat{C} is smooth outside p , C is singular in q if and only if \hat{C} contains both \hat{q}_1 and \hat{q}_2 . If \hat{C} has no tangency with \hat{s} in these two points, its tangent lines in these points correspond to a line in Π_1 and a line in Π_2 different from s . These two lines form the tangent cone of C in q , giving (i).

If \hat{C} is tangent to \hat{s} in \hat{q}_2 , but not in \hat{q}_1 , the tangent cone of C in q is the union of s and a line in Π_1 . Blowing up q , C intersects E_q in s_q and another point in t_1 . Blowing up s , C intersects $E_q \cap E_s$ (which is the fiber over s_q in E_s) in one of the two points intersected by V , that is, $(t_1)_s$ or $(t_2)_s$.

But the tangent line of \hat{C} in \hat{q}_2 is a limit of lines through \hat{q}_2 different from \hat{s} . These lines correspond to lines in Π_2 and, after blowing up q and s , to lines intersecting E_q in t_2 , with limit point $(t_2)_s$. Therefore C intersects $E_q \cap E_s$ in $(t_2)_s$, giving (ii).

If \hat{C} is tangent to \hat{s} in both \hat{q}_1 and \hat{q}_2 , C has a double point in q with tangent cone $2s$. After blowing up q , C has a double point in s_q . The reasoning made above shows that blowing up s , C intersects $E_q \cap E_s$ in $(t_1)_s$ and $(t_2)_s$, proving (iii).

Now suppose q is q_1^s or q_2^s and let \hat{q} be the preimage of q via $\bar{\tau}$. Assume first that C is singular in q . This happens if and only if \hat{C} is tangent to \hat{s} in \hat{q} . The order of the tangency determines the type of the singularity of C , which is always a double point.

If the contact of \hat{C} with \hat{s} is simple, it is defined locally by the contact of a conic and \hat{s} . Since such conic intersects \hat{s} in $2\hat{q}$, it belongs to $\Pi_{X,x}$. Then it is mapped to a plane section of V through q , which is a cuspidal cubic by Lemma 3.2.2. So in this case the tangent cone of C is a double line different from s .

We will now prove that if \hat{C} has a higher order contact with \hat{s} in \hat{q} , then the tangent cone of C is also a double line. For that, suppose that the other components of C_6 (which are lines of the ruling of V , by Corollary 3.1.5) do not contain q . There is no harm in making this assumption, since we could just define C'_6 satisfying it and proceed with the proof for this new curve.

By Lemma 3.2.7 (and further remarks), there is a pencil of cubic surfaces containing C_6 , so let S be a cubic surface containing C_6 and having transversal intersection with V in q , that is:

$$\text{mult}_q(S \cap V) = (\text{mult}_q S) \cdot (\text{mult}_q V)$$

Then, by (3.1) and by the assumption made above, S has multiplicity two in q . In particular, the tangent cone $\mathcal{C}_q S$ has degree two. Lemma 1.1.3 implies:

$$\mathcal{C}_q(S \cap V) = \mathcal{C}_q S \cap \mathcal{C}_q V$$

Since $2s$ is part of the intersection $S \cap V$, it is a component of $\mathcal{C}_q(S \cap V)$. But $\mathcal{C}_q V$ is a double plane, so $\mathcal{C}_q(S \cap V)$ is the union of $2s$ and a double line. By (3.1), this double line is the tangent cone of C_6 in q . Since $C \subset C_6$ has multiplicity two in q , this is the tangent cone of C in q and the assertion is proved.

Finally, suppose C is smooth in $q \in \{q_1^s, q_2^s\}$. Let ℓ be the line of the ruling of V through q .

Since C is smooth in q , the tangent line of \hat{C} in \hat{q} is not s and, by Bezout's theorem, it is not the line through p and \hat{q} (\hat{C} has degree $d - 1$ and multiplicity $d - 2$ in p). Then this tangent line is mapped to a conic in V .

The union of this conic and ℓ forms a plane section of V , so the conic cuts ℓ in two points, one of them being q . But the other point must also be q , since the strict transform of ℓ via $\bar{\tau}$ intersects the tangent line of \hat{C} only in \hat{q} . Therefore the conic is tangent to ℓ in q , and so is C . □

We can now conclude the study of singularities in $C_6 \cap s$.

Proposition 3.2.9. *Let q be a singularity of C_6 in s . Then, blowing up q , E_q is contracted to a triple point of X if and only if one of the following holds:*

- (a) q is a singularity of an irreducible reduced component of C_6 ;
- (b) there are two distinct lines of the ruling through q contained in C_6 ;
- (c) q is q_1^s or q_2^s and C_6 contains the line of the ruling through q ;
- (d) C_6 contains a line of the ruling through q and a curve intersecting this line in two points;
- (e) s is contained in C_6 .

If this is not the case, then \mathcal{X}_q has a fixed curve, which is mapped to a double point of X .

Proof. Let $q \in s$ be a singular point of C_6 . Then it is either a singular point of an irreducible reduced component of C_6 , a point lying on two or more components of C_6 or a point of a non reduced component. We will reorganize these possibilities in the following way:

- (i) q satisfies (a), (b) or (e);
- (ii) q does not satisfy (i) and is the intersection of two or more distinct components of C_6 ;
- (iii) q does not satisfy (i) or (ii) and lies on a non reduced component of C_6 .

By Corollary 3.1.5, if q satisfies (ii), it is the intersection of a (possibly multiple) line and a curve smooth in q , since (a) and (b) are excluded from (ii). By Lemma 3.2.6, this possibility fits in (c), (d) or in:

- (f) C_6 contains a line of the ruling through q , a curve intersecting this line in q with multiplicity one and no other component through q .

Now, if q satisfies (iii), it fits either in (c) or in:

- (g) q is not q_1^s or q_2^s , a line of the ruling through q is a non reduced component of C_6 and no other component contains q .

The next step is to prove that E_q or a curve in it is mapped to a point in X and in which cases each occur.

The situation of case (a) is described in Lemma 3.2.8. If q is not q_1^s or q_2^s , the blow up at q gives $V_q = t_1 + t_2$, a pair of lines. Since \mathcal{X} has multiplicity four in s and two in C_6 , by Lemma 3.2.8, \mathcal{X}_q has multiplicity four in s_q and multiplicity two in two points of E_q , one of them in t_1 and the other in t_2 . These points can be infinitely near to s_q . Then $\mathcal{X}_q = 2t_1 + 2t_2$.

If q is q_1^s or q_2^s , by Lemma 3.2.8 the tangent cone of C_6 in q is a double line. Blow up q and set $V_q = 2t$. Then C_6 is either tangent to E_q or has a double point in it, that is, $(C_6)_q$ is a double point. Then \mathcal{X}_q has multiplicity four in s_q and in $(C_6)_q$. Therefore $\mathcal{X}_q = 4t$, since both s and C_6 lie on V .

This proves that, in case (a), E_q is mapped to a point.

Now suppose q satisfies (b). Each of the two lines of the ruling lies on one of the planes of the tangent cone of V in q . Blowing up q , \mathcal{X}_q has multiplicity four in s_q and multiplicity two in two points, these three being non collinear. Hence it consists of two fixed double lines, and E_q is mapped to a point.

In case (c), let ℓ be the line of the ruling through q . Then either ℓ is a non reduced component of C_6 or it is a simple line in C_6 and there is another component C containing q .

First, suppose ℓ is a double line, that is, $C_6 = 2\ell + C_4$. The case in which it has multiplicity greater than two is analogous. Then there is a line ℓ' infinitely near to ℓ , determined by the surface V .

Let Ω be a general plane containing ℓ . This plane intersects \mathcal{X} in 2ℓ and degree five curves. These curves have a double point in $q = s \cap \Omega$ and three double points in the intersections of C_4 with Ω outside ℓ . There is a fifth double point corresponding to the line ℓ' . Since Ω intersects V in ℓ and a conic, which is tangent to ℓ in q by Lemma 3.2.8, this fifth double point is infinitely near to q , corresponding to the direction of ℓ .

Blow up q . Now \mathcal{X} intersects Ω in 2ℓ and curves having a double point in $E_q \cap \ell$. Therefore \mathcal{X} has multiplicity four in ℓ_q . Since it also has multiplicity four in s_q , it follows that $\mathcal{X}_q = 4t$, where $t = V_q$ is the line through ℓ_q and s_q . Hence E_q is mapped to a point.

Now suppose that ℓ is a simple line in C_6 and there is another component C containing q . If C is singular in q , it fits in (a). If it is smooth in q , then by Lemma 3.2.8 C is tangent to ℓ in q . Blowing up q , C and ℓ intersect E_q in the same point.

Repeating the reasoning made above, \mathcal{X} intersects, after the blow up at q , a general plane Ω through q in 2ℓ and curves having a double point in ℓ_q . Therefore, \mathcal{X} has multiplicity four in s_q and ℓ_q , which implies that \mathcal{X}_q is a fixed line with multiplicity four. Then E_q is mapped to a point.

If q satisfies (d), then by Lemma 3.2.8 q is not q_1^s or q_2^s , and by Lemma 3.2.6 E_q is mapped to a point. This same Lemma asserts that in case (f), \mathcal{X}_q has a fixed double line, which is mapped to a point.

In case (e), q can be any point in s . As seen in the Proof of Proposition 3.2.3, the blow up at s gives:

$$\mathcal{X}_s \equiv 2s' + \{(1, 0)\} \equiv (3, 4)$$

Then s is mapped to a line L . Note that the fiber over q in E_s intersects the double line $s' \equiv (1, 2)$ in two points, which coincide if q is q_1^s or q_2^s . Then blowing up q (instead of blowing up s), \mathcal{X}_q has multiplicity four in s_q and multiplicity two in two points infinitely near to s_q . Therefore \mathcal{X}_q is a pair of (possibly coincident) fixed double lines and E_q is mapped to a point in L .

Finally, suppose q satisfies (g). Again suppose ℓ is a double line, the other cases being analogous. Then $C_6 = 2\ell + C_4$. Let Π be the plane containing ℓ and s . See Figure 3.2 for a sketch of the blow ups that will now be done.

Let Ω be a general plane containing ℓ . Then $V \cap \Omega$ consists of ℓ and a conic intersecting this line in two distinct points. This follows from Lemma 3.2.2 and the fact that q is not q_1^s or q_2^s . On the other hand, $\mathcal{X} \cap \Omega$ consists of 2ℓ and degree five curves with three double points in the intersections with C_4 not lying on ℓ , a double point in q and a fifth double point. This last point must lie on the conic $V \cap \Omega$. And since it corresponds to a line infinitely near to ℓ , it must lie on ℓ . Then this point is not infinitely near to q , otherwise the conic would be tangent to ℓ .

Blowing up q , \mathcal{X}_q has multiplicity four in s_q and a double point in the intersection of ℓ with E_q . There is a second double point infinitely near to it, since there is another double line of \mathcal{X} infinitely near to ℓ . But both double points lie on the line t defined by ℓ_q and s_q .

Note that, unlike case (c), \mathcal{X} does not have multiplicity four in ℓ_q . This follows from the fact that the degree five curves of $\mathcal{X} \cap \Omega$ do not have a

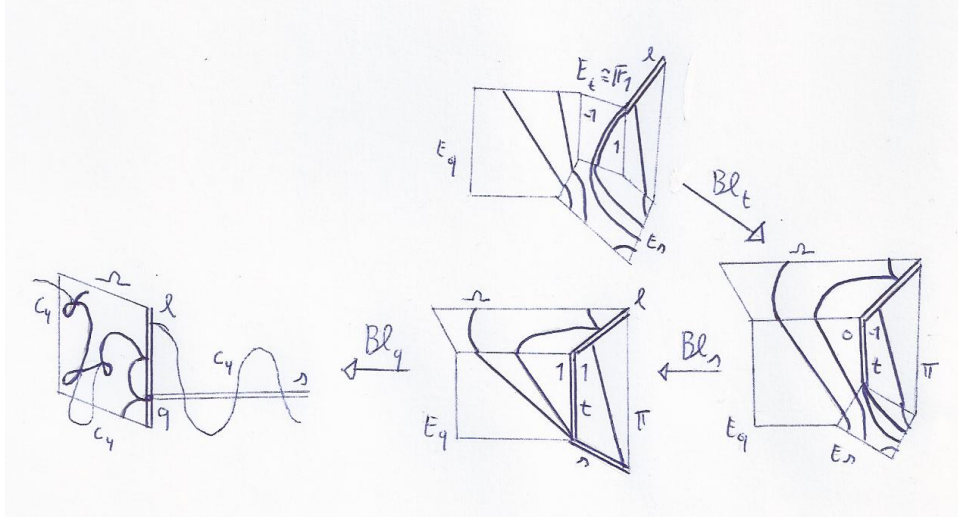


Figure 3.2: \mathcal{X} and the blow ups in case (g) of Proposition 3.2.9

double point infinitely near to q , as noted above. This gives:

$$\mathcal{X}_q = 2t + \{\text{pairs of lines through } s_q\}$$

and E_q is mapped to a conic. Note that $t = \Pi_q$, since Π contains both s and ℓ . Then, after blowing up s , t has normal bundle:

$$N_t = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

Blowing up t gives $(E_q)_t \equiv e_1$. Since \mathcal{X}_t does not intersect E_q , $\mathcal{X}_t \equiv 2e_1 + 2f_1$. It has two or more infinitely near double points (associated to ℓ) lying on $V_t \equiv e_1 + f_1$. Then \mathcal{X}_t is a double curve and E_t is mapped to a point.

It remains to prove that in cases (a) to (e) the image x_q of E_q is a triple point; and that in cases (f) and (g) the image x_q of the fixed double line in E_q is a double point of X . This will be done using Lemma 1.3.3. Let Ω be a general plane through q . The plane Ω is cut by \mathcal{X} in degree seven curves having, among other base points, multiplicity four in q . Since C_6 is singular in q , these curves have two or more double points infinitely near to q .

Blowing up q , in cases (a) to (e) \mathcal{X} has two double points in $e = E_q \cap \Omega$. These double points represent the tangent cone of V in q and are infinitely near if q is q_1^s or q_2^s .

Blowing up these two points, one gets $e^2 = -3$ in Ω having no intersections with \mathcal{X} . Therefore it is mapped to a triple point in the image of Ω . By Lemma 1.3.3, x_q is a triple point of X .

In cases (f) and (g), \mathcal{X} intersects e in one fixed double point and two moving points. This fixed point is the intersection of e with the fixed double line in E_q mentioned above. Blow up the double point and let e' be the exceptional divisor in Ω . Then \mathcal{X} intersects e' in a fixed double point. Blowing it up, $(e')^2 = -2$ in Ω , so it is contracted to the double point x_q in the image of Ω . By Lemma 1.3.3, it is a double point of X . \square

Note that part (e) of this Lemma and Proposition 3.2.3 give the following important result:

Corollary 3.2.10. *Suppose that s is contained in C_6 . Then the image L of s via σ is a triple line of X .*

Suppose now that C_6 has a non reduced component. By Corollary 3.1.5, it must be either s or a line of the ruling of V . If C_6 contains a multiple line ℓ , then there are other lines infinitely near to ℓ lying on the singular locus of \mathcal{X} . We will now study what type of singularities such lines produce on X .

Lemma 3.2.11. *Let ℓ be a line of the ruling of V contained in C_6 with multiplicity $d \geq 2$, that is, $\ell = \ell_1 \prec \ell_2 \prec \dots \prec \ell_d$ are double lines of \mathcal{X} . Then $\ell_1, \dots, \ell_{d-1}$ are mapped to double lines $R = R_1, \dots, R_{d-1}$ of X , with $R_i \prec R_{i+1}$ and ℓ_d is mapped to a cubic scroll through x , with R being a line of its ruling.*

Moreover, if $q = \ell \cap s$ is mapped to a triple point of X , then this point lies on R , unless $d = 2$ and q is q_1^s or q_2^s .

Proof. Denote by Π the plane containing ℓ and s . Then:

$$\mathcal{X} \cap \Pi = 2\ell + 4s + \{\text{lines}\} \quad (3.2)$$

The blow ups that will now be done are illustrated, when $d = 2$, in Figure 3.3. Start blowing up s . After this, ℓ is the complete intersection of V and Π and has normal bundle:

$$N_\ell = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

So blowing up ℓ gives $E_\ell \cong \mathbb{F}_1$ and $V_\ell \equiv e_1 + f_1$. Then ℓ_2 is the intersection of E_ℓ with V . By (3.2) and since \mathcal{X} has multiplicity two in ℓ , it follows that:

$$\mathcal{X}_\ell \equiv 2e_1 + 3f_1 \equiv 2\ell_2 + \{f_1\} \quad (3.3)$$

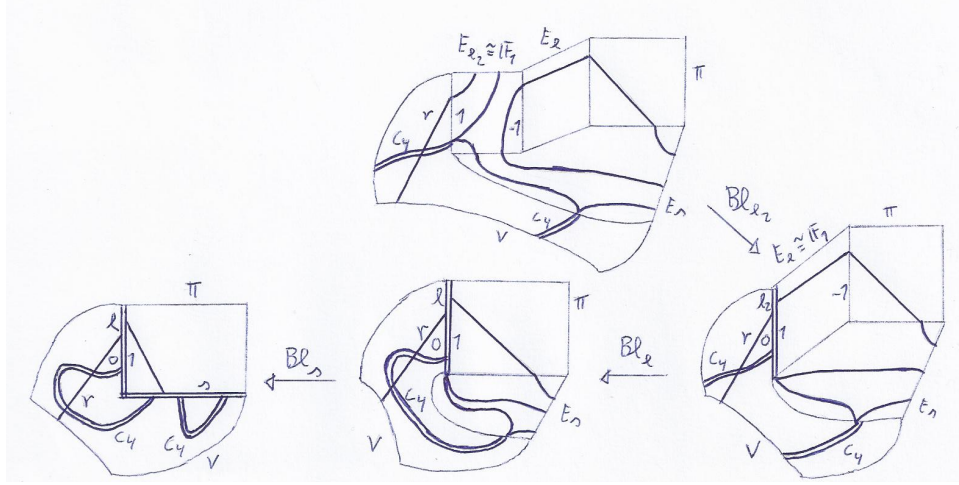


Figure 3.3: The Blow ups for $d = 2$

where $\{f_1\}$ represents the moving part of \mathcal{X}_ℓ , consisting of fibers of E_ℓ . This moving part maps E_ℓ to a line R in X .

The line $\ell_2 = E_\ell \cap V$ has normal bundle:

$$N_{\ell_2} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

Blow up ℓ_2 . If $d > 2$, \mathcal{X} has multiplicity two in $\ell_3 = V_{\ell_2} \equiv e_1 + f_1$ and its intersection with E_{ℓ_2} is the same as the one given in (3.3). Hence E_{ℓ_2} is mapped to a line R_2 . It is infinitely near to R , since ℓ_2 was infinitely near to ℓ . And this result repeats for ℓ_i , with $i \leq d-1$, giving $R_{i-1} \prec R_i$.

After blowing up $\ell_1, \dots, \ell_{d-1}$, blow up ℓ_d , giving again in $E_{\ell_d} \cong \mathbb{F}_1$, $E_{\ell_{d-1}} \cap E_{\ell_d} \equiv e_1$ and $V_{\ell_d} \equiv e_1 + f_1$. Then \mathcal{X}_{ℓ_d} cuts the section e_1 in one point, so:

$$\mathcal{X}_{\ell_d} \equiv 2e_1 + 3f_1$$

having one simple point in r_{ℓ_d} and one double point in the intersection with the other component of C_6 . Remember that, by Corollary 3.1.5, this other component can be r itself, $s' \subset E_s$ or an irreducible curve of degree greater than one.

Therefore \mathcal{X} maps E_{ℓ_d} to a cubic scroll $S(1,2)$ through x . Note that this scroll contains the line R_{d-1} , image of e_1 . It is a line of the ruling not containing x . But R_{d-1} is not a proper line in X , since it is infinitely near to R_{d-2} , so the scroll contains the lines:

$$R = R_1 \prec R_2 \prec \dots \prec R_{d-1}$$

Hence R is a line of the ruling of the cubic scroll.

The next step is to prove, using Lemma 1.3.3, that these are double lines of X . So let p be a general point of ℓ , which is mapped to a general point P of R , and let Ω be a general plane in \mathbb{P}^3 through p .

The plane Ω is cut by \mathcal{X} in degree seven curves having, among other base points, d double points $p = p_1 \prec \cdots \prec p_d$. Blowing up the base points gives a chain of exceptional divisors:

$$E_1 \prec E_2 \prec \cdots \prec E_d$$

with $E_i^2 = -2$ for $i = 1, \dots, d-1$ having no intersection with \mathcal{X} , and $E_d^2 = -1$ being intersected by \mathcal{X} in two moving points. These are mapped to a singular point of X of type A_{d-1} and a conic. The singular point is P itself, hence the general tangent hyperplane section of X through P has double points in $P = P_1 \prec \cdots \prec P_{d-1}$. By Lemma 1.3.3, P is a double point of X .

Note that $P_i \in R_i$. In fact, if Ω is a general plane through $q \in \ell$, then blowing up s and ℓ gives $\Omega_\ell \equiv f_1$. Therefore, in each E_{ℓ_i} , Ω_{ℓ_i} is mapped to a point $P_i \in R_i$. This proves the first part of the Lemma.

Now suppose that $q = l \cap s$ is mapped to a triple point of X , that is, E_q is contracted to a point (see Proposition 3.2.9). Then after the blow up at s , the fiber f^q over q in E_s contains the point ℓ_s . Since we have blown up s , it is f^q which is contracted to a triple point.

Note that V_s is tangent to f^q if and only if q is q_1^s or q_2^s , since the tangent cone of V in these points is a double plane. In this case, the point of tangency is ℓ_s .

Suppose q is q_1^s or q_2^s and $d = 2$. Blowing up ℓ , f^q intersects E_ℓ in a point of $\ell_2 = V_\ell$. Blowing up ℓ_2 , E_{ℓ_2} is mapped to a cubic scroll and E_ℓ is contracted to a line of its ruling, namely R . In this case, f^q does not intersect either V or E_ℓ and its image is not contained in R .

Now suppose that q is q_1^s or q_2^s and $d > 2$. As it was just remarked, blowing up s and ℓ , f^q intersects E_ℓ in a point of $\ell_2 = V_\ell$. After these blow ups, f^q and V_q intersect transversally. Since $d > 2$, blowing up ℓ_2 gives $E_{\ell_2} \cong \mathbb{F}_1$ and:

$$\mathcal{X}_{\ell_2} \equiv 2e_1 + 3f_1 \equiv 2\ell_3 + \{f_1\}$$

where $\{f_1\}$ is the moving part of \mathcal{X}_2 , mapping E_{ℓ_2} to R_2 . After this blow up, f^q and V_q do not intersect, which implies that $f^q \cap E_{\ell_2}$ does not lie on ℓ_3 . Therefore the image of f^q is contained in R_2 .

Finally, suppose q is not q_1^s or q_2^s . Then, after the blow up at s , V_s is not

tangent to f^q . Therefore, blowing up ℓ , f^q no longer intersects V_ℓ and:

$$\mathcal{X}_\ell \equiv 2e_1 + 3f_1 \equiv 2\ell_2 + \{f_1\}$$

Since the point $f^q \cap E_\ell$ does not lie on $\ell_2 = V_\ell$, it is mapped to a point of R . □

3.2.3 A family of surfaces in X

Lemma 3.2.7 and the remarks after it assert that there is a cubic surface, other than V , containing C_6 and that any cubic surface containing C_6 must also contain the lines r and s . So let \mathcal{Q} be the linear system of cubic surfaces in \mathbb{P}^3 containing C_6 , r and s . Then by (3.1) \mathcal{Q} has fixed intersection with V . Suppose that s is not contained in C_6 , as this case will be considered in Lemma 3.2.14.

The exact sequence:

$$0 \rightarrow \mathcal{Q} - V \rightarrow \mathcal{Q} \rightarrow \mathcal{Q}|_V \rightarrow 0$$

gives:

$$0 \rightarrow H^0(\mathbb{P}^3, \mathcal{Q} - V) \rightarrow H^0(\mathbb{P}^3, \mathcal{Q}) \rightarrow H^0(\mathbb{P}^3, \mathcal{Q}|_V)$$

where the last map is clearly surjective. Since $H^0(\mathbb{P}^3, \mathcal{Q} - V) \cong \mathbb{C}$ and \mathcal{Q} has fixed intersection with V , it follows that \mathcal{Q} is a pencil.

Since \mathcal{Q} is generated by V and another cubic, its base locus is given by (3.1), that is, $C_6 + r + 2s$. This means that the surfaces in \mathcal{Q} have in common a curve infinitely near to s . It also implies that if s is not contained in C_6 , V is the only surface in \mathcal{Q} having a singular curve.

Note that if $r \subset C_6$, the surfaces in \mathcal{Q} contain five lines of the ruling, which implies they contain r and r' (see Lemma 3.1.4).

The following Lemma gives a rational representation of a cubic in \mathcal{Q} that will be used next:

Lemma 3.2.12. *Suppose $s \not\subset C_6$ and let S_3 be a cubic in \mathcal{Q} different from V . Then there is a map*

$$\phi : \mathbb{P}^2 \dashrightarrow S_3$$

defined by cubic curves through six (possibly infinitely near) points, such that the line r is the exceptional line of the blow up at a point in \mathbb{P}^2 and s is the image of a conic through the other five points.

Proof. Since $s \not\subset C_6$ and S_3 is not V , S_3 is not ruled. Fix a point $p \in S_3$ not lying on a line of S_3 and let \mathcal{L} be the linear system of quadric surfaces through p containing r and s . These quadrics must also contain the line ℓ_p through p intersecting both r and s . Therefore the base locus of \mathcal{L} is a curve of type $(1, 2)$ in each quadric and \mathcal{L} has projective dimension two. Then it defines a map:

$$\psi : S_3 \dashrightarrow \mathbb{P}^2$$

To show that it is birational, take a point $P \in \mathbb{P}^2$. Its preimage via ψ is the set of points in the intersection of S_3 with two general quadrics of \mathcal{L} not lying on the base locus of \mathcal{L} . Two quadrics of \mathcal{L} intersect in r , s , ℓ_p and a line ℓ' skew to ℓ_p . Since $\ell_p \not\subset S_3$, this line intersects S_3 in p , a point in r and a point in s , all of them lying on the base locus of \mathcal{L} . The line ℓ' intersects S_3 in a point in r , a point in s and one further point. Then the preimage of P is one point and ψ is birational.

Let \mathcal{L}' be the linear system in \mathbb{P}^2 defining the inverse of ψ . It is defined by quartic curves. Indeed, the residual intersection of a quadric of \mathcal{L} with S_3 is a quartic curve through p of type $(3, 1)$ in this quadric. Then a general plane section of S_3 intersects this curve in four points not lying on the base locus. These four points are mapped by ψ to the intersection of a curve in \mathcal{L}' with a line. This shows that \mathcal{L}' is defined by quartic curves.

The map ψ contracts any line in S_3 intersecting r and s , since such line has fixed intersection with \mathcal{L} . If S_3 is a smooth cubic, there are five such lines and \mathcal{L}' has five simple base points. If S_3 is singular, some of these base points may be infinitely near.

Two other curves are contracted by ψ . The residual intersection of S_3 with the plane spanned by p and r is a conic intersecting each quadric of \mathcal{L} in p , two points in r and one point in s . A similar conic can be defined taking the plane through p containing s . These conics are contracted by ψ to double points of \mathcal{L}' .

Therefore \mathcal{L}' consists of quartic curves having two double points and four (possibly infinitely near) simple points. There are no further base points, since the degree of the image surface is:

$$4^2 - 2 \cdot 4 - 5 \cdot 1 = 3$$

Let us now investigate the image of r and s via ψ .

Blow up the line r . Then $\mathcal{L}_r \equiv (1, 1)$ and the only base point of these curves is $\ell_p \cap E_r$. On the other hand, $(S_3)_r \equiv (2, 1)$ does not contain this point. Therefore r is mapped by ψ to a cubic curve. It must contain the five base points of \mathcal{L}' , since these are contractions of lines intersecting $(S_3)_r$. It

also contains both double points, but it has multiplicity two in one of them, namely the contraction of the conic intersecting r in two points.

The same reasoning applies to s . Therefore r and s are mapped to cubic curves C^r and C^s through all simple points having multiplicity one in a double point of \mathcal{L}' and multiplicity two in the other double point.

Now apply in \mathbb{P}^2 a standard quadratic transformation T_1 centered in the two double points and one simple point of \mathcal{L}' . Then \mathcal{L}' is mapped to a linear system \mathcal{L}'_1 of cubic curves with six base points. The curves C^r and C^s are mapped to conics C_1^r and C_1^s through five of the six base points. Let P_1 be the base point in $C_1^r \setminus C_1^s$.

Now apply a second quadratic transformation T_2 centered in P_1 and in two base points of \mathcal{L}'_1 in $C_1^r \cap C_1^s$. Then \mathcal{L}'_1 is mapped to a linear system \mathcal{L}'_2 of cubics with six base points. The image of C_1^r via T_2 is a line C_2^r through two base points P_2 and Q_2 . The image of C_1^s is a conic C_2^s containing five of the base points, including P_2 and Q_2 .

Finally apply a third quadratic transformation T_3 centered in P_2 , Q_2 and in the base point of \mathcal{L}'_2 not lying on C_2^s . Again \mathcal{L}'_2 is mapped to a linear system \mathcal{L}'_3 of cubics with six base points. Now the image of C_2^r is a base point of \mathcal{L}'_3 and the image of C_2^s is a conic through the other five points.

Hence \mathcal{L}'_3 defines the map ϕ described in the statement. □

Lemma 3.2.13. *Suppose C_6 does not contain s . Then the pencil \mathcal{Q} is mapped by σ to a pencil \mathcal{Q}' of weak Del Pezzo quartic surfaces in X . The base locus of \mathcal{Q}' is a line L .*

Note that the surface $V \in \mathcal{Q}$ is contracted to x . But it has an extra multiplicity in s , so it corresponds to the image of s , that is, $D_4^x \in \mathcal{Q}'$.

Proof. Let S_3 be a cubic in \mathcal{Q} different from V . Then it is normal and, since r and s are disjoint, it is not a cone. Hence it is weak Del Pezzo cubic (see [Do1]).

Let $\phi : \mathbb{P}^2 \dashrightarrow S_3$ be the birational map given in Lemma 3.2.12, that is, ϕ is defined by a linear system of cubics through six points, such that the line r is the exceptional line of the blow up at a point $p_1 \in \mathbb{P}^2$ and s is the image of a conic through the other five base points.

In \mathbb{P}^2 , write (d, m, n) for the class of a curve with degree d having multiplicity m at $p_1 = \phi^{-1}(r)$ and n at each of the other five points. As usual, $(0, -1, 0)$ stands for the exceptional divisor of the blow up at p_1 , whereas the line s is mapped to a $(2, 0, 1)$ curve. A plane section of S_3 is mapped to a curve of type $(3, 1, 1)$.

We will adopt this notation for classes of curves in S_3 , considering their birational image, via ϕ , in \mathbb{P}^2 . Then:

$$(9, 3, 3) \equiv V \cap S_3 \equiv C_6 + r + 2s \equiv (d, m, n) + (0, -1, 0) + (4, 0, 2)$$

which implies that C_6 is a curve of type $(5, 4, 1)$ in S_3 .

Now, intersect S_3 with \mathcal{X} :

$$(21, 7, 7) \equiv 2C_6 + r + 4s + F_4 \equiv (10, 8, 2) + (0, -1, 0) + (8, 0, 4) + (d, m, n)$$

So the moving part F_4 of $\mathcal{X} \cap S_3$ consists of curves of type $(3, 0, 1)$, which are mapped via ϕ^{-1} to cubics through five points in \mathbb{P}^2 . This linear system of cubics maps \mathbb{P}^2 to a weak Del Pezzo surface of degree four. This surface is the image of S_3 via σ .

To prove the assertion on the base locus, remember that the surfaces in \mathcal{Q} have a curve infinitely near to s in common. Then, after the blow up at the base locus of \mathcal{X} , the fixed part of \mathcal{Q}_s is the only curve in the base locus of \mathcal{Q} . Therefore, its image is contained in the base locus of \mathcal{Q}' . But the moving part is mapped to a moving intersection of surfaces in \mathcal{Q}' with D_4^x , the image of s . Since $D_4^x \in \mathcal{Q}'$, it follows that there is no such moving part, that is, \mathcal{Q}_s is a fixed curve.

Consider the blow up at s . Then \mathcal{Q}_s is a fixed curve t_L of type $(2, 1)$. It contains the five points corresponding to the intersections with C_6 . Remember that $\mathcal{X}_s = (3, 4)$ and it has double points in these five points. Then \mathcal{Q}_s is mapped to a curve of degree:

$$(2, 1) \cdot (3, 4) - 5 \cdot 2 = 1$$

and this line L lies on the base locus of \mathcal{Q}' .

Moreover, some surfaces in \mathbb{P}^3 (including exceptional divisors of these blow ups) are contracted by σ to curves or points. These are V , E_r (or $E_{r'}$ if $r \subset C_6$) and E_ℓ if ℓ is a multiple line in C_6 . Depending on the intersection of \mathcal{Q} with these surfaces, their images can be contained in the base locus of \mathcal{Q}' .

After the blow up at the base locus of \mathcal{X} , the intersection of \mathcal{Q} with V consists of \mathcal{Q}_s , which was already analysed. The intersection with E_r is $\mathcal{Q}_r \equiv (2, 1)$ and it contains the four double points of $\mathcal{X}_r \equiv (6, 1)$, so each surface of \mathcal{Q}' intersects the image ℓ_x of E_r in one point.

If $r \subset C_6$, $\mathcal{X}_r \equiv (5, 2) \equiv r' + (3, 1)$ and $\mathcal{Q}_r = r'$. The moving part intersects r' in five fixed points, corresponding to the five lines of C_6 . Blowing up these five lines and r' gives $\mathcal{X}_{r'} \equiv \mathcal{Q}_{r'} \equiv (0, 1)$, so again each surface intersects the image of r' in one point (see the proof of Proposition 3.2.3).

If ℓ is a multiple line, let Π be the plane containing ℓ and s . Then:

$$\mathcal{Q} \cap \Pi = \ell + s + \{\text{lines}\}$$

Blowing up s and ℓ gives $\mathcal{X}_\ell \equiv 2e_1 + 3f_1 \equiv 2\ell' + \{f_1\}$ (see Lemma 3.2.11) and $\mathcal{Q}_\ell \equiv e_1 + 2f_1 \equiv \ell' + \{f_1\}$. Therefore, each surface of \mathcal{Q}' cuts the image of E_ℓ in one point.

Hence L is the base locus of \mathcal{Q}' . The second part of the Lemma follows from Proposition 3.2.3. \square

The case in which s is contained in C_6 is now analysed:

Lemma 3.2.14. *Suppose $s \subset C_6$, so that $C_6 = \ell_1 + \dots + \ell_4 + 2s$. Define \mathcal{Q} to be the linear system of cubic surfaces containing ℓ_1, \dots, ℓ_4, r and having multiplicity two in s . Then \mathcal{Q} is a pencil and it is mapped by σ to a pencil \mathcal{Q}' of quartic surfaces in X . These surfaces are projections of Veronese quartic surfaces and have a double line L , the image of s , which is the base locus of \mathcal{Q}' .*

Note that the surface $V \in \mathcal{Q}$ is contracted to x . But it contains s' , so it corresponds to the image of s' , that is, $D_4^x \in \mathcal{Q}'$.

Proof. For the images of s and s' , see Proposition 3.2.3.

Let S be a cubic of \mathcal{Q} different from V , then:

$$S \cap V = \ell_1 + \dots + \ell_4 + r + 4s \tag{3.4}$$

which implies that \mathcal{Q} has fixed intersection with V . From the exact sequence:

$$0 \rightarrow \mathcal{Q} - V \rightarrow \mathcal{Q} \rightarrow \mathcal{Q}|_V \rightarrow 0$$

we have:

$$0 \rightarrow H^0(\mathbb{P}^3, \mathcal{Q} - V) \rightarrow H^0(\mathbb{P}^3, \mathcal{Q}) \rightarrow H^0(\mathbb{P}^3, \mathcal{Q}|_V)$$

where the last map is clearly surjective. Since $H^0(\mathbb{P}^3, \mathcal{Q} - V) \cong \mathbb{C}$ and $H^0(\mathbb{P}^3, \mathcal{Q}|_V) \cong \mathbb{C}$, it follows that \mathcal{Q} is a pencil.

The cubic S is a non normal cubic of type (i) in Proposition 3.1.1, since it has a double line s and contains a line r skew to s . The conics in S give a plane representation similar to the one of V via $\bar{\sigma}$, in which $r \equiv (0, -1)$ and $2s \equiv (1, 0)$.

By (3.4), S does not contain s' . The intersection of S with \mathcal{X} gives:

$$\sum_{i=1}^4 2\ell_i + r + 8s + F_4 \equiv (14, 7) \equiv \sum_{i=1}^4 2 \cdot (1, 1) + (0, -1) + 4 \cdot (1, 0) + (a, b)$$

which implies that $F_4 \equiv (2, 0)$, which does not contain s' .

Now note that the intersection of F_4 with s is actually a double point of F_4 . Indeed, blowing up s , $S_s \equiv (1, 2)$ intersects $V_s = s' \equiv (1, 2)$ in four points, each being the intersection of a line ℓ_i with E_s . And S_s intersects \mathcal{X}_s , which is the union of $2s'$ and moving fibers, in these four points and two moving points. Then these two points are the intersection of F_4 with E_s . Since these points lie on a fiber over a point of s , the assertion follows.

Since $F_4 \equiv (2, 0)$, the moving part of $\mathcal{X} \cap S$ consists of curves birationally equivalent to a linear system of conics in \mathbb{P}^2 with no base points. This linear system is not complete, since these conics intersect the image of s in \mathbb{P}^2 in pairs of points that are sent to the same point: the double point of F_4 in s . Therefore, S is mapped to a projection of a Veronese quartic surface, having double line L , the image of s .

The base locus of \mathcal{Q} is given by (3.4). After blowing up the base locus of \mathcal{X} , \mathcal{Q} no longer has base curves. Then we must look for surfaces being contracted by σ , which are V , E_r , E_s and E_ℓ if ℓ is a multiple line in C_6 .

After the blow ups, \mathcal{Q} has no intersection with V . And as in the previous Lemma, $\mathcal{Q}_r \equiv (2, 1)$ and the surfaces of \mathcal{Q}' intersect ℓ_x in moving points.

In E_s , $\mathcal{Q}_s \equiv (1, 2)$ containing the four points determined by the lines ℓ_i . As seen in the proof of Proposition 3.2.3, $s' \equiv (1, 2)$ and:

$$\mathcal{X}_s = 2s' + \{\text{moving fibers}\} \equiv (3, 4)$$

Then the curves \mathcal{Q}_s intersect s' in those four points and the moving part of \mathcal{X}_s maps each of these curves to L .

Now suppose ℓ is a multiple line in C_6 . If Π is the plane containing ℓ and s , then:

$$\mathcal{Q} \cap \Pi = \ell + 2s$$

As in the previous Lemma, blowing up s and ℓ gives, in $E_\ell \cong \mathbb{F}_1$, $\Pi_\ell \equiv e_1 + f_1$. But now $\mathcal{Q}_\ell = \ell' \equiv e_1 + f_1$. Then the surfaces of \mathcal{Q}' do not intersect the image of E_ℓ . So one of these surfaces must contain this line.

In conclusion, the base locus of \mathcal{Q}' is L .

□

Remember that by Corollary 3.2.10, if s is contained in C_6 , it is mapped to a triple line L . And we have just proved that all surfaces in \mathcal{Q}' are singular along L in this case. Here is a slight generalization of these facts:

Proposition 3.2.15. *The triple points of X , if any, lie on the base line L of \mathcal{Q}' . Moreover, the surfaces of \mathcal{Q}' have multiplicity two in these points.*

Proof. If $s \subset C_6$, it is mapped to L , which is a triple line in this case. Other triple points arise either from the intersection of two lines in C_6 or from the intersection of s with a multiple line in C_6 through q_1^s or q_2^s (see Proposition 3.2.9 and Corollary 3.1.5). Let q be this point. Blowing up s , in both cases the triple point is the contraction of the fiber over q . By Proposition 3.2.3:

$$\mathcal{X}_s \equiv 2s' + \{(1, 0)\}$$

which implies that the triple point lies on L . By Lemma 3.2.14, L is a double line of \mathcal{Q}' , so the second assertions is also proved in this case.

Now suppose that C_6 does not contain s . By Proposition 3.2.9, a triple point of X is the image of the exceptional divisor E_q of the blow up at a singular point q of C_6 lying on s . As explained in the proof of Lemma 3.2.13, L is the image of a curve t_L of type $(2, 1)$ in the exceptional divisor E_s of the blow up at s . This curve contains the five intersections of C_6 with s .

Let $q \in \mathbb{P}^3$ be a point that is mapped to x_q , a triple point of X , and let t^q be the fiber over q in E_s . Then q fits in one of the cases (a) to (d) of Proposition 3.2.9. It will be proven that t_L contains t^q .

If q is not q_1^s or q_2^s , then the tangent cone of C_6 in q have components in both planes Π_1 and Π_2 , which form the tangent cone of V in q . Then the curve t_L intersects t^q in two points and, therefore, contains this fiber.

If q is q_1^s or q_2^s , then V_s is tangent to t_q . Since q is a singular point of C_6 , t_L intersects V_s in a point of t^q and another point infinitely near to it. Since V_s is tangent to t^q , both points lie on this line, which implies that t_L contains t^q .

Therefore, we have that:

$$\mathcal{Q}_s = t_L = t^q + t'_L \equiv (1, 0) + (1, 1)$$

where t'_L intersects t^q in one point. Then x_q lies on L .

We will now prove that x_q is a double point of D_4^x , the surface of \mathcal{Q}' through x . Since x is a general point of X , x_q is a double point of surfaces of \mathcal{Q}' .

Remember that D_4^x is the image of E_s via σ and, since x_q is a triple point of X , it is the image of t^q , the fiber over q in E_s . Therefore, the linear system $\mathcal{X}_s \equiv (3, 4)$ has two (possibly infinitely near) double points in t^q . Blowing up these double points, we get $(t^q)^2 = -2$ in E_s . Then its image, x_q , is a double point of D_4^x .

□

The pencil \mathcal{Q}' will be used to give a geometrical description of the three-fold X . For that we need another result:

Lemma 3.2.16. *There are at least seven independent quadrics of \mathbb{P}^7 containing X .*

Proof. The exact sequence:

$$0 \rightarrow \mathcal{I}_{X/\mathbb{P}^7}(2) \rightarrow \mathcal{O}_{\mathbb{P}^7}(2) \rightarrow \mathcal{O}_X(2) \rightarrow 0$$

gives the cohomology exact sequence:

$$0 \rightarrow H^0(\mathcal{I}_{X/\mathbb{P}^7}(2)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^7}(2)) \rightarrow H^0(\mathcal{O}_X(2)) \rightarrow H^1(\mathcal{I}_{X/\mathbb{P}^7}(2)) \rightarrow 0 \quad (3.5)$$

We know that $H^0(\mathcal{O}_{\mathbb{P}^7}(2))$ stands for the linear system of quadric hypersurfaces of \mathbb{P}^7 , giving $h^0(\mathcal{O}_{\mathbb{P}^7}(2)) = \binom{9}{2} = 36$.

The quadric sections of X correspond, via τ , to the linear system $\mathcal{L} = 2\mathcal{X}$ of surfaces of degree 14 having multiplicity eight along s , multiplicity four along C_6 and multiplicity two along r . Therefore $h^0(\mathcal{O}_X(2)) = h^0(\mathcal{L})$.

Let $\mathcal{L} - V$ be the linear system of surfaces S of degree 11 such that $S + V \in \mathcal{L}$, that is, having multiplicity six along s , multiplicity three along C_6 and containing r . Define analogously $\mathcal{L} - 2V$, $\mathcal{L} - 3V$ and $\mathcal{L} - 4V$.

For $k = 0, 1, 2, 3$, the restriction of $\mathcal{L} - kV$ to V gives the exact sequence:

$$0 \rightarrow \mathcal{L} - (k+1)V \rightarrow \mathcal{L} - kV \rightarrow (\mathcal{L} - kV)|_V \rightarrow 0$$

Passing to cohomology, we have:

$$0 \rightarrow H^0(\mathcal{L} - (k+1)V) \rightarrow H^0(\mathcal{L} - kV) \rightarrow H^0((\mathcal{L} - kV)|_V) \rightarrow 0 \quad (3.6)$$

where the exactness on the right follows from the surjectiveness of the last map.

Note first that \mathcal{L} has fixed intersection with V , namely:

$$\mathcal{L}|_V = 16s + 4C_6 + 2r$$

Therefore $h^0(\mathcal{L}|_V) = 1$.

Next, the fixed part of $(\mathcal{L} - V)|_V$ is $12s + 3C_6 + r$. Using the notation of page 58 for curves in V , the moving part of $(\mathcal{L} - V)|_V$ is of type:

$$(22, 11) - (6, 0) - (15, 12) - (0, -1) = (1, 0)$$

Then it corresponds to lines in \mathbb{P}^2 , giving $h^0((\mathcal{L} - V)|_V) = 3$.

The linear system $\mathcal{L} - 2V$ consists of degree eight surfaces having multiplicity four along s and two along C_6 . Then the fixed part of $(\mathcal{L} - 2V)|_V$ is $8s + 2C_6$ and the class of its moving part is:

$$(16, 8) - (4, 0) - (10, 8) = (2, 0)$$

This gives a base point free linear system of conics in \mathbb{P}^2 , so $h^0((\mathcal{L} - 2V)|_V) = 6$.

Finally, $\mathcal{L} - 3V$ consists of degree five surfaces having multiplicity two along s and containing C_6 . The fixed part of $(\mathcal{L} - 3V)|_V$ is $4s + C_6$ and the moving part is of type:

$$(10, 5) - (2, 0) - (5, 4) = (3, 1)$$

Then it corresponds to cubic curves in \mathbb{P}^2 with one base point. Therefore $h^0((\mathcal{L} - 3V)|_V) = 9$.

On the other hand, $\mathcal{L} - 4V$ is the linear system of quadric surfaces in \mathbb{P}^3 , giving $h^0(\mathcal{L} - 4V) = \binom{5}{3} = 10$. Then (3.6) gives, for $k = 3$:

$$h^0(\mathcal{L} - 3V) = h^0(\mathcal{L} - 4V) + h^0((\mathcal{L} - 3V)|_V) = 10 + 9 = 19$$

Applying the same equation with $k = 2$, gives:

$$h^0(\mathcal{L} - 2V) = h^0(\mathcal{L} - 3V) + h^0((\mathcal{L} - 2V)|_V) = 19 + 6 = 25$$

Then, for $k = 1$ we have:

$$h^0(\mathcal{L} - V) = h^0(\mathcal{L} - 2V) + h^0((\mathcal{L} - V)|_V) = 25 + 3 = 28$$

And finally, for $k = 0$:

$$h^0(\mathcal{L}) = h^0(\mathcal{L} - V) + h^0(\mathcal{L}|_V) = 28 + 1 = 29$$

Returning to (3.5), we have:

$$\begin{aligned} h^0(\mathcal{I}_{X/\mathbb{P}^7}(2)) &= h^0(\mathcal{O}_{\mathbb{P}^7}(2)) - h^0(\mathcal{O}_X(2)) + h^1(\mathcal{I}_{X/\mathbb{P}^7}(2)) \\ &= 36 - 29 + h^1(\mathcal{I}_{X/\mathbb{P}^7}(2)) \\ &\geq 7 \end{aligned}$$

□

Proposition 3.2.17. *Suppose that the linear system \mathcal{X} is relatively complete. Then there is a cone F over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ with vertex L , such that X is the residual intersection of F with two quadrics containing a \mathbb{P}^4 of its ruling.*

In particular, X is an OADP threefold.

Proof. According to Remark 1.4.3, X is linearly normal. Then, by Theorem 1.4.4, X lies on a rational normal scroll F , described by the \mathbb{P}^4 's spanned by the quartics in \mathcal{Q}' . These \mathbb{P}^4 's have a line in common, namely L . Therefore F is a cone with vertex L over a scroll Y in \mathbb{P}^5 . This scroll contains a one-dimensional family of disjoint planes, so Y is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. In particular, F is the intersection of three quadrics containing it.

Note that X cuts a \mathbb{P}^4 of F in a quartic surface. Indeed, if C_6 does not contain s , then by Lemma 3.2.13 this intersection is a weak Del Pezzo surface containing L . If s is contained in C_6 it is a projection of a Veronese quartic surface from an outside point and has multiplicity two along L .

Let $\eta : \text{Bl}_L(\mathbb{P}^7) \rightarrow \mathbb{P}^7$ be the blow up of L , let G be the strict transform of F and E be the intersection of G and the exceptional divisor. Note that E has dimension three. In G , set H_0 for the class of the strict transform of a hyperplane section of F and H_1 for the class of the strict transform of a \mathbb{P}^4 of the ruling. Note that $H_1^2 = 0$.

By Lemma 3.2.16, there are at least seven independent quadrics containing X . Three of these can be chosen containing F . Then there are at least four other quadrics through X which do not contain F . Among these, pick two: Q_1 and Q_2 . Note that $Q_1 \cap Q_2$ intersects Π , a general \mathbb{P}^4 of the ruling of F , in a quartic surface that coincides with $X \cap \Pi$. If $s \subset C_6$, this surface has multiplicity two along L .

Let Q'_1 and Q'_2 be the total transforms via η of $Q_1 \cap F$ and $Q_2 \cap F$, and let X' be the strict transform of $X \subset F$.

The divisors Q'_1 and Q'_2 contain both X' and E . So let S be residual intersection, that is:

$$Q'_1 \cap Q'_2 = X' + E + S$$

If s is contained in C_6 , then $Q_1 \cap Q_2$ has multiplicity two along L . In this case write:

$$Q'_1 \cap Q'_2 = X' + 2E + S$$

In both cases, S does not contain E . Since X intersects a \mathbb{P}^4 of the ruling of F in a quartic surface, it follows that S consists of threefolds contained in rulings of G . Moreover:

$$\deg(\eta(S)) = \deg(Q_1 \cap Q_2 \cap F) - \deg(X) = 12 - 8 = 4$$

These considerations imply that $S \equiv 4H_0H_1$. Therefore:

$$X' + \delta E \equiv (2H_0)^2 - 4H_0H_1 \equiv (2H_0 - H_1)^2$$

with $\delta \in \{1, 2\}$. Taking the image via η in \mathbb{P}^7 , it follows that X is the residual intersection of F with two quadrics containing a \mathbb{P}^4 of the ruling.

The proof that X is an OADP threefold is the same done in [CMR, Example 2.6].

□

3.3 Description of the general OADP threefold of degree 8

This section is dedicated to the study of the case in which the curve C_6 of Proposition 3.1.3 is general, that is, it is smooth. This defines the general OADP threefold of degree 8. A small description of this threefold is now given using the inverse of its general tangential projection.

Proposition 3.3.1. *Let X be the threefold corresponding to the case in which C_6 is smooth. Then X is the smooth scroll in lines given in Example 1.4.2.*

The threefold X contains a pencil \mathcal{Q}' of Del Pezzo quartic surfaces with base locus a line L . The surface $D_4^x \subset \mathcal{Q}'$ through x is the image of the line s . The line r is mapped to the line through x

The curve C_6 in the base locus of \mathcal{X} is mapped to a degree 16 scroll in conics S_{16}^x having multiplicity 5 in x . It intersects quartics of \mathcal{Q}' in curves of degree 10. Its intersection with the quartic of \mathcal{Q}' through x consists of five conics through this point.

Through x there is a one-dimensional family of conics parametrized by C_6 . These conics describe the scroll S_{16}^x . Besides these, there are other five conics through x , which lie on D_4^x .

Proof. Set $C = C_6$. Since C is smooth, by Lemma 3.2.4 the associated variety X is smooth, and by Proposition 3.2.17, it is an OADP threefold. Since it has degree eight, by the classification of smooth OADP threefolds done in [CMR], it follows that X is projectively equivalent to the variety given in Example 1.4.2.

Note that the fact that X is ruled by lines also follows from Corollary 3.1.6. Moreover, the description of X as an intersection of divisors of a cone F over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ with vertex L , given in Example 1.4.2, is also obtained in Proposition 3.2.17.

The images of r and s are given by Proposition 3.2.3 and the assertions on the pencil \mathcal{Q}' are explained in Lemma 3.2.13. The fact that the general surface in \mathcal{Q}' is smooth follows from the description of X as an intersection of F with two general quadric hypersurfaces containing a \mathbb{P}^4 of the ruling.

Next, we study the image of C via σ . Remember that $C \equiv (5, 4)$, that is, it is the image via $\bar{\tau}$ of a degree five curve in $E \cong \mathbb{P}^2$ having multiplicity four in p , the base point of $\Pi_{X,x}$. In particular, C is rational. The inclusions $C \subset V \subset \mathbb{P}^3$ give an exact sequence:

$$0 \rightarrow \mathcal{N}_{C/V} \rightarrow \mathcal{N}_{C/\mathbb{P}^3} \rightarrow \mathcal{N}_{V/\mathbb{P}^3|_C} \rightarrow 0$$

Since C is of type $(5, 4)$ in V , $C^2 = 9$ and $\mathcal{N}_{C/V} = \mathcal{O}_{\mathbb{P}^1}(9)$. On the other hand, V is a cubic hypersurface, so $\mathcal{N}_{V/\mathbb{P}^3|_C} = \mathcal{O}_{\mathbb{P}^1}(3)$. Now, since $\mathcal{N}_{C/\mathbb{P}^3}$ is a line bundle over a rational curve, it splits and:

$$\mathcal{N}_{C/\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^1}(9) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$$

Therefore, blowing up C gives $E_C \cong \mathbb{F}_6$. The surface V has multiplicity one along C , except for the five points in s , where the multiplicity is two. Then:

$$V_C \equiv e_6 + af_6 \equiv F + e_6 + (a - 5)f_6$$

where $F \equiv 5f_6$ is the union of the fibers over the five points in s .

Let S be a general surface in \mathcal{Q} . Remember that $V \in \mathcal{Q}$ and that:

$$V \cap S = C + r + 2s$$

This implies that $S_C \equiv e_6 + af_6$ and that it intersects V_C in four simple points corresponding to $r \cap C$ and five double points corresponding to $s \cap C$. Then:

$$4 + 5 \cdot 2 = V_C \cdot S_C = -6 + 2a$$

and this gives $a = 10$. Set $\widehat{V}_C = V_C - F \equiv e_6 + 5f_6$.

On the other hand, \mathcal{X} has multiplicity two along C , except for the points in s , where it is four. Then:

$$\mathcal{X}_C \equiv 2F + 2e_6 + bf_6$$

Let $\widehat{\mathcal{X}}_C$ be the moving part. It has multiplicity two in the points s_C and multiplicity one in r_C . We know that $\widehat{\mathcal{X}}_C$ has fixed intersection with \widehat{V}_C , which is smooth in these points. Then:

$$4 + 5 \cdot 2 = \widehat{\mathcal{X}}_C \cdot \widehat{V}_C = -12 + 10 + b$$

which gives $b = 16$. Note that $\widehat{\mathcal{X}}_C$ has no moving intersection with F .

Finally, the degree of the image of E_C via σ is:

$$(2e_6 + 16f_6)^2 - 4 - 5 \cdot 4 = -24 + 64 - 24 = 16$$

The curve \widehat{V}_C is contracted to the point x .

Note that $(\widehat{V}_C)^2 = 4$ and it contains nine base points of $\widehat{\mathcal{X}}_C$. Then S_{16}^x has multiplicity five in x .

The moving intersection of $\widehat{\mathcal{X}}_C$ with S_C is:

$$(2e_6 + 16f_6)(e_6 + 10f_6) - 4 - 2 \cdot 5 = -12 + 20 + 16 - 14 = 10$$

So S_{16}^x intersects a general surface of \mathcal{Q}' in a degree 10 curve. Since s_C consists of the five double points of $\widehat{\mathcal{X}}_C$, S_{16}^x intersects D_4^x in five conics through x .

To determine all conics through x , note that the tangential projection of such conic is a point. Therefore this conic must lie on the image of the exceptional divisors of \mathcal{X} .

A general fiber of E_C intersects \mathcal{X}_C in two moving points and intersects V_C in a point outside the base locus of \mathcal{X}_C . This gives a one-dimensional family of conics through x , which describes S_{16}^x and is parametrized by C .

On the other hand, the Del Pezzo surface D_4^x , image of E_s , contains ten conics through x . As noted above, five of these conics lie on S_{16}^x .

□

3.4 Description of a singular example

In this section, we will describe an example where the curve C_6 is the union of a conic and four lines. Two of these lines are infinitely near, that is, it is a double line of C_6 . In V , this means that we are writing:

$$(5, 4) = (1, 0) + (1, 1) + (1, 1) + (2, 2)$$

Suppose that the two simple lines ℓ_1 and ℓ_2 intersect in a point q_1 (that must lie on s). Suppose also that the point of intersection of C and s is q_1 . Let ℓ_3 be the other line and ℓ_4 be the line infinitely near to it.

Throughout this section, let Σ be the plane containing C , let Π_i be the plane containing s and ℓ_i , for $i = 1, 2, 3$, and let Γ be the plane containing ℓ_1, ℓ_2 and r . Then the tangent cone of V in q_1 is the union of Π_1 and Π_2 . By Lemma 3.2.6, C intersects one of the lines through q_1 , say ℓ_2 , in a second point q_2 . In particular, ℓ_2 lies on Σ and the tangent line to C at q_1 is $\Sigma \cap \Pi_1$. Set $q_3 = \ell_3 \cap C$ and suppose $\ell_3 \cap s$ is not q_1^s or q_2^s .

All the results of this section refer to this specific configuration. Figure 3.4 illustrates the base locus of \mathcal{X} .

The point q_1 is an intersection of three components of C_6 . The following remark will be useful throughout this section:

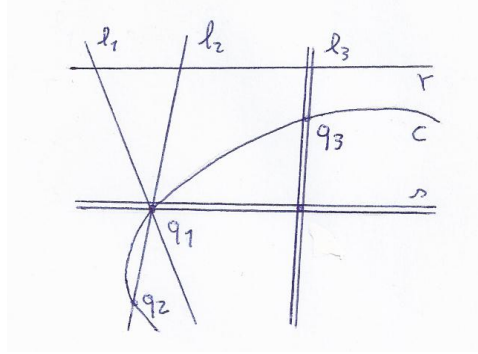


Figure 3.4: The base locus of \mathcal{X} in Section 3.4.

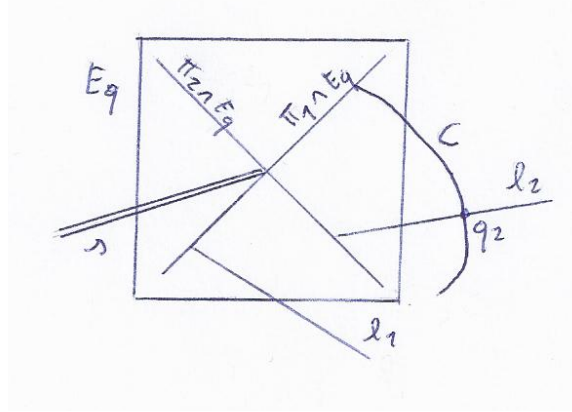


Figure 3.5: The blow up at $q = q_1$

Remark 3.4.1. Consider the blow up at $q = q_1$. As seen in the proof of Lemma 3.2.6, \mathcal{X}_q is a pair of double lines intersecting in s_q . One of these lines, namely $(\Pi_1)_q$, contains the intersections of E_q with C and l_1 ; while the other line $(\Pi_2)_q$ contains the intersection with l_2 . See Figure 3.5.

3.4.1 Singularities of X

By Lemma 3.2.4, the singularities of X are images of singularities of C_6 . Proposition 3.2.9 describes the image of q_1 and asserts that $l_3 \cap s$ is not mapped to a triple point. The images of q_2 and q_3 are explained in Proposition 3.2.5. The double line l_3 is studied in Lemma 3.2.11. This gives:

Proposition 3.4.2. *The points q_i are mapped to points x_i in X , for $i = 1, 2, 3$, and ℓ_3 is mapped to a line R . The point x_1 is a triple point, x_2 and x_3 are double points and R is a double line of X . Among the three singular points, only x_3 lies on R . Moreover, the projectivization of the tangent cone of X in x_1 is the union of a smooth quadric surface and a plane.*

Proof. Only the two last assertions need a proof.

The points q_1 and q_2 do not lie on ℓ_3 , so their images do not lie on R . And as noticed in the proof of Lemma 3.2.11, after the blow up at s and ℓ_3 the moving part of \mathcal{X}_{ℓ_3} consists of fibers over points. Since $q_3 \notin s$, it is mapped to a point in R .

To prove the assertion on the tangent cone, let $\widehat{\mathcal{X}}$ be the linear system of surfaces in \mathcal{X} having multiplicity five in $q = q_1$. Then, blowing up q , the degree five curves \mathcal{X}_q have multiplicity four in s_q and multiplicity two in C_q , $(\ell_1)_q$ and $(\ell_2)_q$. By Remark 3.4.1, the two former lie on $t_1 = (\Pi_1)_q$, and the latter lies on $t_2 = (\Pi_2)_q$. Then:

$$\mathcal{X}_q = 2t_1 + t_2 + \{\text{conics}\}$$

where the moving part has two base points: $(\ell_2)_q$ and s_q . These conics map E_q to a smooth quadric surface.

By Lemma 1.1.4, the projectivization of the tangent cone of X in x_1 contains a smooth quadric surface. Since x_1 is a point of multiplicity three, the result follows. \square

By Proposition 3.2.15, x_1 is a double point of the degree four surfaces in \mathcal{Q}' . As noted in the proof of this result, the blow up at s gives:

$$\mathcal{Q}_s = t^q + t'_L \equiv (2, 1) \tag{3.7}$$

where t^q is the fiber over q_1 and $t'_L \equiv (1, 1)$ is the curve containing $\ell_1 \cap E_s$, $\ell_3 \cap E_s$ and $\ell_4 \cap E_s$ (which is infinitely near to $\ell_3 \cap E_s$).

3.4.2 The image of C_6

We will now study the images of the irreducible components of C_6 via σ .

Lemma 3.4.3. *Let $i \in \{1, 2, 4\}$. The line ℓ_i is mapped to a cubic scroll through x . This scroll also contains the point x_i , if $i = 1, 2$. If $i = 4$, it*

contains x_3 and R is a line of its ruling. It intersects quartics of \mathcal{Q}' in conics through a fixed point. This point is x_1 when $i = 1, 2$. It lies on L also when $i = 4$.

Proof. For $i = 1, 2$, set $\ell = \ell_i$, $\Pi = \Pi_i$ and $q = q_1 = s \cap \ell$. The intersection of \mathcal{X} with Π is $2\ell + 4s$ and moving lines. Let Ω be a plane through ℓ . Then \mathcal{X} cuts Ω in 2ℓ and quintics intersecting ℓ in the double point q and three moving points. If $i = 1$, there is another base point infinitely near to q , but it does not lie on ℓ .

First blow up s , to avoid fixed components. By Remark 3.4.1, \mathcal{X}_s intersects the fiber over q in two fixed double points. One of them is $(\ell_2)_s$, while the other contains the intersection of E_s with ℓ_1 and C .

At this stage, $l = \Omega \cap \Pi$, where at Ω it had a point blown up. So:

$$N_\ell = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

Blowing up l gives $E_\ell \cong \mathbb{F}_1$, $\Omega_\ell \equiv e_1 + f_1$ and $\Pi_\ell \equiv e_1$. Hence:

$$\mathcal{X}_\ell \equiv 2e_1 + 3f_1$$

with one double point in C_ℓ and one simple point in r_ℓ . Note that there is no other base point, since the blow up at s has separated ℓ_1 and C from ℓ_2 in q .

Applying Lemma 1.1.1, this linear system is birationally equivalent, in \mathbb{P}^2 , to cubics with one double and two simple points. After a standard quadratic transformation, these correspond to conics with one base point. Hence E_ℓ is mapped to a cubic scroll of type $S(1, 2)$.

The section $V_\ell \equiv e_1 + f_1$ containing the base points is contracted to x and the fiber through the double point C_ℓ is contracted to x_i . The lines of the ruling of the scroll are the images of sections $e_1 + f_1$ through C_ℓ . The fiber through r_ℓ is mapped to the directrix line and r_ℓ is mapped to the line of the ruling through x .

A cubic of \mathcal{Q} intersects Ω in ℓ and a conic, and intersects Π in ℓ, s and a line. Blowing up s gives, by (3.7), $\mathcal{Q}_s = t^q + t'_L$, which is a fixed curve. Next, blowing up ℓ gives $\mathcal{Q}_\ell \equiv e_1 + 2f_1$.

We'll now study separately the cases $i = 1$ and $i = 2$. Both are represented in Figure 3.6, as well as case $i = 4$.

If $i = 2$, \mathcal{Q}_{ℓ_2} has three base points. One of them lies on the intersection with E_s , namely, it is $(t^q)_{\ell_2}$. Since it is the fiber in E_s over q_1 , this base point is mapped to x_1 . Note that $(t^q)_{\ell_2}$ is not a base point of \mathcal{X}_{ℓ_2} .

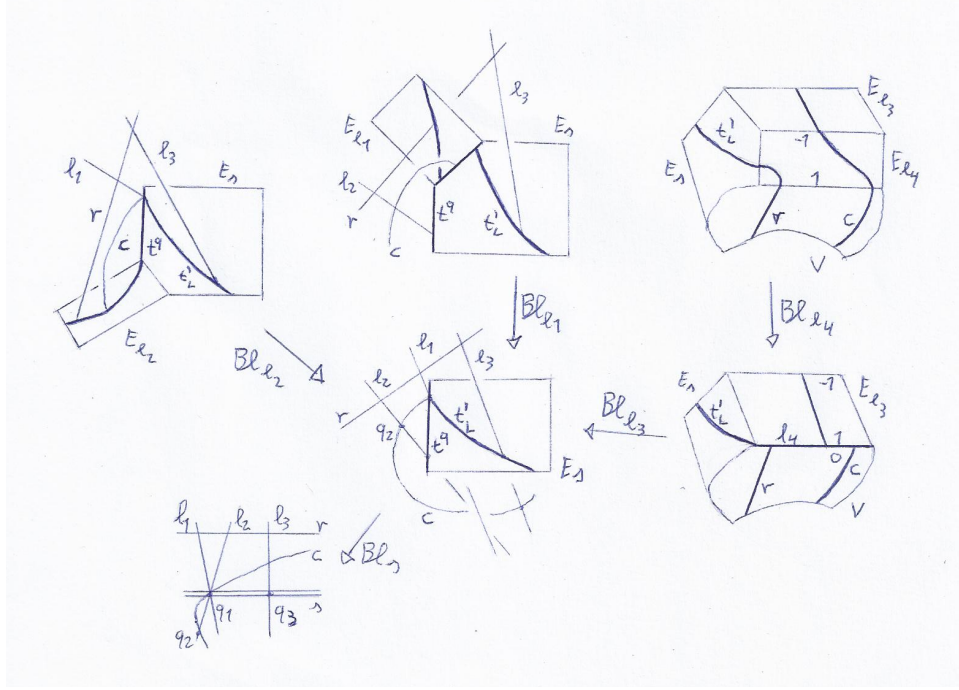


Figure 3.6: \mathcal{Q} and the blow ups in ℓ_i

The second is r_{ℓ_2} , a simple point of \mathcal{X}_{ℓ_2} ; and the third is C_{ℓ_2} , a double point of \mathcal{X}_{ℓ_2} . There are no other base points, since $(\mathcal{Q}_{\ell_2})^2 = 3$. Therefore, the cubic scroll intersects a quartic of \mathcal{Q}' in a curve of degree:

$$(e_1 + 2f_1)(2e_1 + 3f_1) - 1 - 2 = 2$$

These conics intersect each other in x_1 . Indeed, two of the three base points of \mathcal{Q}_{ℓ_2} are also base points of \mathcal{X}_{ℓ_2} , while the other one is mapped to x_1 .

If $i = 1$, three of the base points of \mathcal{Q}_{ℓ_1} (namely $(t^q)_{\ell_1}$, $(t'_L)_{\ell_1}$ and C_{ℓ_1}) lie on $(E_s)_{\ell_1} \equiv f_1$. Then this is a fixed component and it is mapped to x_1 . The moving part of \mathcal{Q}_{ℓ_1} is of type $e_1 + f_1$ and it has a simple base point in r_{ℓ_1} . Then the degree of its image is:

$$(e_1 + f_1)(2e_1 + 3f_1) - 1 = 2$$

As before, these conics intersect each other in x_1 .

This finishes the analysis for $i = 1, 2$.

For $i = 4$, after blowing up s , ℓ_3 and $\ell = \ell_4$ we also have that $\mathcal{X}_{\ell} \equiv 2e_1 + 3f_1$. This is explained in the proof of Lemma 3.2.11. In the same

Lemma it is showed that R is a line of the ruling of the scroll, image of ℓ . Moreover:

$$\mathcal{Q}_{\ell_3} \equiv e_1 + 2f_1 \equiv \ell_4 + \{f_1\}$$

and $\mathcal{Q}_\ell \equiv e_1 + 2f_1$ as before. These have base points in $(t'_L)_\ell, r_\ell$ and C_ℓ and are mapped to conics through a fixed point of L , since $(t'_L)_\ell$ is not a base point of \mathcal{X}_ℓ . See Figure 3.6.

□

Next, we study the image of C :

Lemma 3.4.4. *The conic C is mapped to a quartic Veronese surface through x, x_1, x_2 and x_3 . It cuts each quartic of \mathcal{Q}' in a conic through x_1 .*

The union of this surface, the plane $\sigma(\Sigma)$ and the cubic scroll, image of ℓ_2 , is a tangent hyperplane section of X at x . The plane intersects the scroll in a line and the Veronese surface in a conic.

Proof. First note that:

$$\mathcal{X} \cap \Sigma = 2\ell_2 + 2C + \{\text{lines}\} \quad (3.8)$$

Next, blow up s and then ℓ_2 . Now, $C = \Sigma \cap V$. In V , $C^2 = 1$, and in Σ , C had one point blown up. Hence:

$$N_C = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

Blow up C , so $E_C \cong \mathbb{F}_2$, $\Sigma_C \equiv e_2$ and $V_C \equiv e_2 + 2f_2$. Since \mathcal{X}_C cuts f_2 in two points, it follows that:

$$\mathcal{X}_C \equiv 2e_2 + 6f_2$$

It has three double points, in the intersections with ℓ_1, ℓ_3 and ℓ_4 (infinitely near to $(\ell_3)_C$).

These curves can be birationally mapped to \mathbb{P}^2 using Lemma 1.1.1. Their images are sextic curves with two double points, a third double point infinitely near to one of these, a point of multiplicity four and a double point infinitely near to it. After two standard quadratic transformations, \mathcal{X}_C is mapped to a linear system of conics with no base points.

Therefore C is mapped to a quartic Veronese surface and V_C is contracted to x . The points x_1 and x_3 are the contractions of the fibers through the respective double points. The point x_2 is the image of the intersection of E_C with the fiber in E_{ℓ_2} over q_2 .

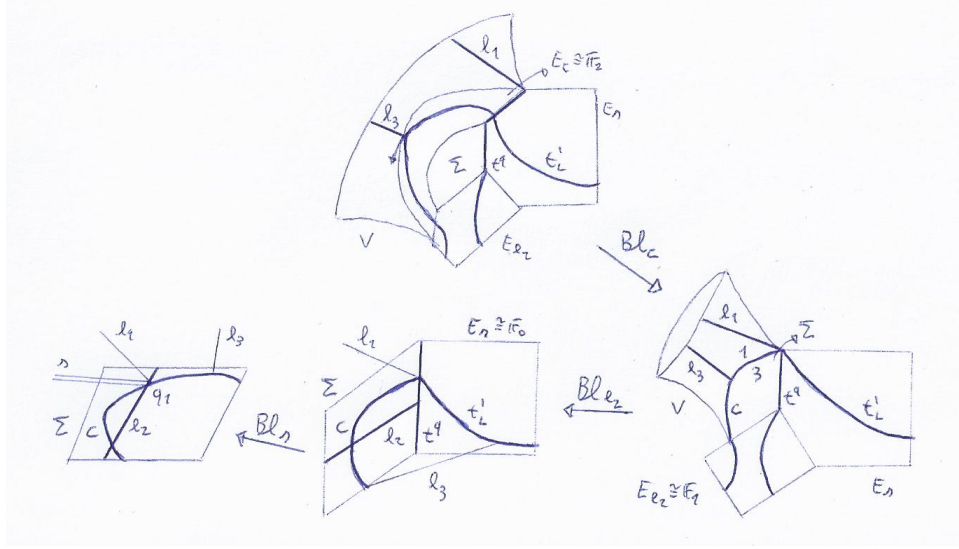


Figure 3.7: \mathcal{Q} and the blow ups in s , ℓ_2 and C

By (3.7), when blowing up s the cubics of \mathcal{Q} cut E_s in two fixed curves t^q and t'_L . The curve t'_L cuts $\Sigma_s = t^q$ in the point C_s .

After blowing up ℓ_2 and C , \mathcal{Q}_C cuts the section $V_C = e_2 + 2f_2$ in three points, namely $(\ell_1)_C$, $(\ell_3)_C$ and $(\ell_4)_C$. But it cuts the fiber f_2^q of E_C over q_1 in three points: $(\ell_1)_C$, $(t^q)_C$ and $(t'_L)_C$. Hence:

$$\mathcal{Q}_C = f_2^q + \{e_2 + 2f_2\}$$

and the moving part has two base points: $(\ell_3)_C$ and $(\ell_4)_C$. It is mapped by \mathcal{X}_C to a conic. The fixed part is mapped to the point x_1 .

Figure 3.7 illustrates \mathcal{Q} under these blow ups.

This proves the first part of the Lemma. The second part follows from Lemma 1.3.2 and from (3.8). \square

Note that $\sigma(\Sigma)$ is not the only plane in X . The planes Π_1 and Π_2 are also mapped to planes, since:

$$\mathcal{X} \cap \Pi_i = 4s + 2\ell_i + \{\text{lines}\}$$

and these moving lines have no base points. For the same reason, the plane containing s and ℓ_3 is mapped to a plane in X .

3.4.3 Description of the example

Putting together the results in this example, we have:

Proposition 3.4.5. *The threefold X has a double line R , a double point x_2 and a triple point x_1 . It contains a pencil \mathcal{Q}' of weak Del Pezzo quartic surfaces having a double point in x_1 . The base locus of this pencil is a line L containing x_1 .*

The tangent cone of X in x_1 is the union of a rank four quadric cone and a \mathbb{P}^3 .

Three lines of the base locus of \mathcal{X} are mapped to cubic scrolls through x and one line is mapped to R . The conic C is mapped to a quartic Veronese surface. Each of these surfaces intersects the quartics of \mathcal{Q}' in conics through a fixed point of L .

In total, counting multiplicities, the curve C_6 is mapped to a reducible surface of degree 16. It has multiplicity 5 in x , multiplicity three in the triple point of X and multiplicity two in the double points, including R . It cuts quartics of \mathcal{Q}' in five conics, three of them through x_1 .

3.5 The Cayley's ruled cubic case

In this Section, suppose V is Cayley's ruled cubic, with equation given in Proposition 3.1.1, (ii). This situation will give us no new varieties, but different projections of varieties having fundamental surface the general cubic (i).

We start by explaining that r is infinitely near to s in V and by computing its image in X .

Lemma 3.5.1. *Blowing up s gives:*

$$V_s = r + s' \equiv (1, 1) + (0, 1)$$

where s' is the strict transform of the plane intersecting V in $3s$.

Proof. By Proposition 3.1.1, V can be given by:

$$x_0^2 x_3 + x_0 x_1 x_2 + x_1^3 = 0 \tag{3.9}$$

Moreover, the map $\bar{\tau}$ can be defined by:

$$(t_0 : t_1 : t_2) \mapsto (t_2^2 : t_1 t_2 : t_0 t_2 - t_1^2 : -t_0 t_1)$$

with $p = (1 : 0 : 0) \in E$. Let s_p be the line $t_2 = 0$ through p .

Blowing up p , the union of s_p and the exceptional line is \hat{s} , which is mapped to s , the double line of V . Then r is infinitely near to s .

We'll compute the blow up at s directly from (3.9), where s is given by $x_0 = x_1 = 0$. Setting $x_0 = uv$, $x_1 = v$, $x_2 = w$ and $x_3 = 1$, (3.9) gives:

$$v^2(u^2 + uw + v) = 0$$

with $v = 0$ being the equation of E_s .

Then, restricting to E_s gives $u(u + w) = 0$. The strict transform of the plane $x_0 = 0$, which intersects V in $3s$, is given by $u = 0$. And the curve $v = u + w = 0$ is of type $(1, 1)$ in E_s . It is the intersection of the strict transform of the quadric $x_0x_3 + x_1x_2 = 0$ with E_s .

To find the equation of r , we blow up E in p . So put $t_0 = 1$, $t_1 = y$ and $t_2 = yz$. Then the resolution of \bar{r} is:

$$(1 : y : yz) \mapsto (y^2z^2 : y^2z : yz - y^2 : -y) = (-yz^2 : -yz : y - z : 1)$$

where r is defined as $y = 0$ in the target, giving $x_0 = x_1 = 0$. The blow up at s done above gives:

$$-yz^2 = uv \quad ; \quad -yz = v \quad ; \quad y - z = w$$

which implies that $y = u + w$. Therefore r is defined infinitely near to s as $u + w = 0$, the $(1, 1)$ curve in E_s . □

Lemma 3.5.2. *The image of $r \in E_s$ is a line ℓ_x through x .*

Proof. Let Q^r be the quadric such that $Q^r \cap E_s = r$. It is smooth, otherwise Q^r would be singular in a point of s , and r would be reducible. But this is not the case, since V_s would contain a fiber of E_s , that is, V would have a triple point: it would be a cone.

Since \mathcal{X} has multiplicity four in s and one in r , it follows that:

$$\mathcal{X} \cap Q^r = 5s + C_9 \equiv (5, 0) + (2, 7)$$

where the equivalence classes refer to curves in the smooth quadric Q^r . Then C_9 intersects s in seven points.

Blowing up s gives $\mathcal{X}_s \equiv (3, 4)$. Since it contains $r \equiv (1, 1)$, the movable part is of type $(2, 3)$ and intersects r in five points. By Proposition 3.1.3, four of these points lie on C_6 and are base points of the movable part of \mathcal{X}_s . They are also base points of $\mathcal{X} \cap Q^r$, so only three of the seven points in $C_9 \cap E_s$ are moving.

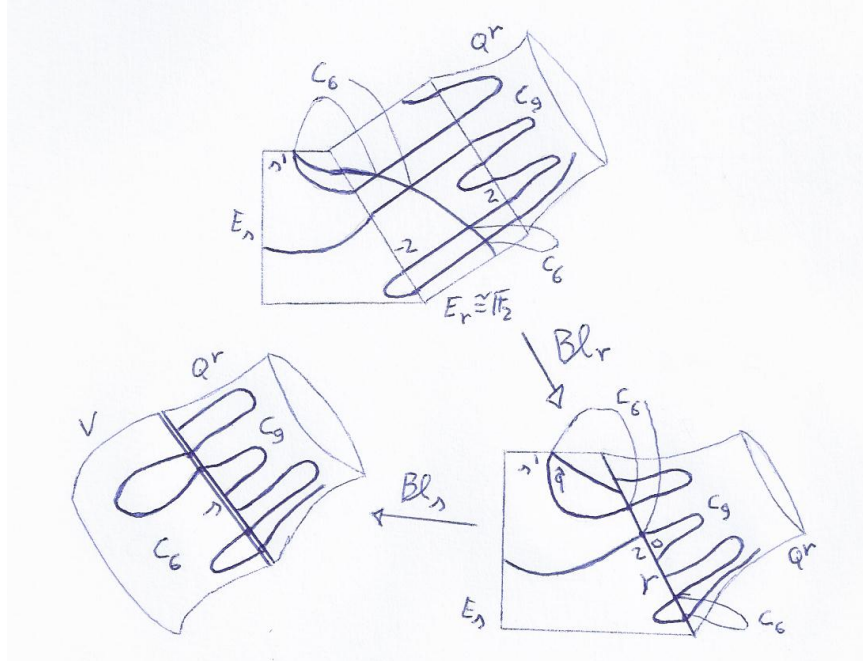


Figure 3.8: \mathcal{X} and the blow ups in s and r

Since $r = Q^r \cap E_s$, its normal bundle is:

$$N_r = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

Therefore, blowing up r gives $(Q^r)_r \equiv e_2 + 2f_2$ and $(E_s)_r \equiv e_2$. The curves of \mathcal{X}_r intersect a general fiber in one point and have four double points, corresponding to the intersections with C_6 . Then \mathcal{X}_r contains four fibers of E_r . And by what was noted above, the moving part of \mathcal{X}_r intersects E_s in one point and Q^r in three points. Then:

$$\mathcal{X}_r \equiv 4f_2 + \{e_2 + 3f_2\}$$

where $4f_2$ is the fixed part, consisting of the fibers over the four base points in r . Each fiber is intersected by C_6 in one point. These are the base points of the moving part.

The blow ups are represented in Figure 3.8.

By Lemma 1.1.1, the moving part is birationally equivalent to cubics in \mathbb{P}^2 with 4 simple points, one double point and another simple point infinitely near to it, which is Cremona equivalent to the linear system of lines with

one base point. Moreover, the fixed fibers have no moving intersection with the moving part. Therefore E_r is mapped to a line.

By Lemma 3.5.1, V_s has two components: r and s' , which intersects r in one point. Then, after the blow up at r , V_r intersects E_s in one point. Then $V_r \equiv e_2 + 3f_2$ and it intersects the moving part of \mathcal{X}_r in its base points. Therefore the image of E_r contains x . □

The next result indicates that the center of the tangential projection may not be a general point:

Lemma 3.5.3. *Suppose V is the Cayley's ruled cubic. Then the line ℓ_x through the point $x \in X$ from which was made the tangential projection intersects another line L of X . Moreover, for a general point $y \in X$, the line ℓ_y through y does not intersect L .*

Proof. By Proposition 3.1.3, C_6 intersects s in five points and r in four points. Blowing up s , by Lemma 3.5.1 r is infinitely near to s of type $(1, 1)$. Then C_6 cuts E_s in four points in r and one point in $s' \equiv (0, 1)$. Let \hat{q} be this point and $t^q \equiv (1, 0)$ be the fiber through it in E_s . Let $q \in C_6 \cap s$ be the point corresponding to this fiber, that is, \hat{q} is infinitely near to q .

As it was already noted, the moving part of \mathcal{X}_s is of type $(2, 3)$. It has a double point in \hat{q} , therefore t^q is mapped to a line, name it L . But t^q intersects r in one point, which is mapped to a point of ℓ_x , by Lemma 3.5.2. Then ℓ_x intersects L .

Given a general point $y \in X$, it is mapped by τ to a general point y' in \mathbb{P}^3 . Since y' is a general point and C_6 cuts s in five points, there is a unique line through y' intersecting C_6 and s in distinct points. For general y' , it does not contain q . Therefore it is mapped to the line ℓ_y through y in X , which does not intersect L . □

The fact that x is not a general point of X will become clear in the next result.

But before proceeding, note that the proof of Lemma 3.2.7 can be repeated to show that there is a pencil of cubic surfaces \mathcal{Q} containing C_6 , r and s , supposing C_6 does not contain s . Let S be a cubic surface in \mathcal{Q} different from V . Then:

$$V \cap S = C_6 + 2s + r$$

which implies that V is the only surface in \mathcal{Q} with a double curve.

Note that S has a double point in q (notation of Lemma 3.5.3). In fact, blowing up q , S_q contains the non collinear points $(C_6)_q$, s_q and r_q infinitely near to s_q , so it is not a line.

In the general setting, S is a general cubic with a point of type A_1 in q . Then it is the image of a \mathbb{P}^2 blown up at six points lying on a conic C_q via the strict transform of the linear system of cubics through these points. The conic C_q is contracted to the double point q and the six points are mapped to six lines through q . Let $p_1 \in \mathbb{P}^2$ be the point mapped to s .

For a curve in S , write (d, m, n) if its strict transform in \mathbb{P}^2 has degree d , multiplicity m in p_1 and multiplicity n in the other five base points. Therefore $s \equiv (0, -1, 0)$ and a general plane section of S is of type $(3, 1, 1)$. The point q is mapped to $C_q \equiv (2, 1, 1)$. Then a plane section through q can be written as:

$$H \cap S \equiv (3, 1, 1) = (2, 1, 1) + (1, 0, 0)$$

Where $(2, 1, 1)$ corresponds to C_q .

Since S contains C_6 , r and s , then:

$$C_6 + 3s = V \cap S \equiv (9, 3, 3) = (4, 2, 2) + (0, -3, 0) + (5, 4, 1)$$

where $(4, 2, 2)$ is the class of $2C_q$ and $(0, -3, 0)$ represents $3s$. Therefore $C_6 \equiv (5, 4, 1)$.

Intersecting with the linear system \mathcal{X} , which has multiplicity four in q , gives:

$$2C_6 + 5s + F_4 = \mathcal{X} \cap S \equiv (21, 7, 7) = (8, 4, 4) + (0, -5, 0) + (13, 8, 3)$$

with $(8, 4, 4)$ corresponding to $4C_q$ and $(0, -5, 0)$ to $5s$. Then the moving part F_4 is of type $(3, 0, 1)$, that is, it corresponds to cubics in \mathbb{P}^2 with five base points.

A similar result can be obtained in the non general setting. If s is contained in C_6 , the same reasoning done in Lemma 3.2.14 shows that the moving part of $\mathcal{X} \cap S$, where S is a general cubic of \mathcal{Q} , is birationally equivalent to a non complete linear system of conics in \mathbb{P}^2 with no base points. Remember that in this case S is singular in s .

These considerations will be used in the proof of the following proposition.

Proposition 3.5.4. *Suppose V is Cayley's ruled cubic. Then there is a Cremona transformation in \mathbb{P}^3 that maps \mathcal{X} to a linear system defining the inverse of a general tangential projection of X , having fundamental surface a general projection of $S(1, 2)$ from an outer point.*

Proof. The linear system \mathcal{X} defines the inverse of $\tau = \tau_x$, the projection of X from $T_x X$. Let y be a general point in X , let τ_y be the projection from $T_y X$. We are looking for the Cremona transformation $T = \tau_y \circ (\tau_x)^{-1}$.

Take a plane in the target \mathbb{P}^3 . It's preimage via τ_y is a tangent hyperplane section of X at y , that is, a hyperplane section of X with a double point in y . This section is mapped by τ_x to a surface of \mathcal{X} having a double point in $y' = \tau_x(y)$.

Therefore, the linear system \mathcal{Y} of surfaces in \mathcal{X} with a double point in y' defines the Cremona transformation T . Note that the line $\ell' = \tau(\ell_y)$ through y' intersecting s and C_6 lies on the base locus of \mathcal{Y} , since it intersects these surfaces in eight points.

Blowing up y' , the linear system $\mathcal{Y}_{y'}$ consists of conics with one base point, the intersection of ℓ' with $E_{y'}$. Therefore it maps $E_{y'}$ to a projection V' in \mathbb{P}^3 of $S(1, 2)$. By Proposition 3.1.1, this projection is not general if and only if the preimage in $E_{y'}$ of the double line of V' contains the base point $(\ell')_{y'}$.

Now consider the cubic S of \mathcal{Q} through y' . By the remarks made above, if s is not contained in C_6 the moving part of $\mathcal{X} \cap S$ corresponds in \mathbb{P}^2 to cubics with five base points. Then the moving part of $\mathcal{Y} \cap S$ corresponds to cubics with five simple and one double points. This linear system maps \mathbb{P}^2 to a line, that is, S is mapped by T to a line.

Note that blowing up the double point, these plane cubics cut the exceptional divisor in a pencil with degree two. Therefore, blowing up y , \mathcal{Y} cuts the line $S \cap E_{y'}$ in a non complete linear system. Hence, the image of S via T is the double line of V' .

Since the base point of $\mathcal{Y}_{y'}$ is $\ell'_{y'}$, V' is not a general projection of $S(1, 2)$ if and only if ℓ' lies on the tangent plane of the cubic surface S in y' . If this happens, then in fact S contains ℓ' .

But S cannot contain ℓ' , since this implies that S is ruled, by the generality of y' . And therefore, S would be a cone or singular along a line. As it was discussed above, S cannot have a singular line. And it cannot be a cone over a smooth cubic, since q is a double point of S not contained in ℓ' . Therefore $E_{y'}$ is mapped to a general projection of $S(1, 2)$.

If $s \subset C_6$, the moving part of $\mathcal{X} \cap S$ is birationally equivalent to a non complete linear system of conics with no base points. Then the moving part of $\mathcal{Y} \cap S$ corresponds to a pencil of conics with one double point. Then S is again mapped to the double line of V' .

In this case, S is singular along s . Moreover $C_6 = 2s + \ell_1 + \dots + \ell_4$. If S contains ℓ' , then there are two lines of the ruling of S intersecting in a point not lying on s . But this is not possible.

□

3.6 Conclusion

An idea of different possibilities of irreducible components of C_6 was given in Corollary 3.1.5. The next Lemma indicates that the total number of different configurations, and hence, of different Bronowski threefolds with cubic fundamental surface is very high.

Lemma 3.6.1. *Suppose that C_6 contains a conic. Then there are 49 different configurations for C_6 .*

Proof. As it was already remarked, C_6 is the union of a conic C and four lines ℓ_1, \dots, ℓ_4 , which can be infinitely near. We divide the configurations in five groups:

- (a) ℓ_1, \dots, ℓ_4 are not infinitely near
- (b) $\ell_1 \prec \ell_2$
- (c) $\ell_1 \prec \ell_2 \prec \ell_3$
- (d) $\ell_1 \prec \ell_2 \prec \ell_3 \prec \ell_4$
- (e) $\ell_1 \prec \ell_2$ and $\ell_3 \prec \ell_4$

By Lemma 3.2.6, the conic C intersects s in one point and intersects a line ℓ of the ruling in one or two points. If C intersects ℓ in a point of s , there are three possibilities: either C intersects ℓ transversally in one point, or it intersects ℓ in s and in a second point outside s (so ℓ and C are coplanar), or it is tangent to ℓ in a point of s (which is q_1^s or q_2^s). If two lines of the ruling intersect in $q \in s$ and this point lies on C , then C intersects one of these lines in a second point not in s .

Start with group (a). If the four lines are skew, either C intersects none of the lines in s , or intersects one of them in one of the three different ways described above. If ℓ_1 and ℓ_2 intersect in q , there are the same four possibilities and also C can contain q . If ℓ_1 intersects ℓ_2 and ℓ_3 intersects ℓ_4 , then there are two possibilities. So group (a) counts $4 + 5 + 2 = 11$ possibilities.

Now consider group (b). In other words, ℓ_1 is a double line in C_6 . If ℓ_1, ℓ_3 and ℓ_4 are skew, there are three possibilities for C intersecting the double line in s , three for a simple line and one to intersect none in s . If ℓ_3 and ℓ_4

intersect, the possibilities are $3 + 1 + 1 = 5$. If ℓ_1 and ℓ_3 intersect, then C can be coplanar to the double line ℓ_1 or to the simple line ℓ_3 , or it intersects s in ℓ_4 or outside the three lines. This gives $2 + 3 + 1 = 6$ possibilities. Therefore, group (b) has $7 + 5 + 6 = 18$ possible configurations.

In group (c), ℓ_1 is a triple line. It can be skew to ℓ_4 , producing $1 + 3 + 3 = 7$ possibilities. If the two lines intersect, there are three possible configurations. So group (c) counts $7 + 3 = 10$ configurations.

In group (d) there is only one line with multiplicity four, which gives four possibilities.

In group (e), the two double lines can be skew or can intersect. If they are skew, there are four options. If they intersect, there are two. So it gives $4 + 2 = 6$ possibilities.

Therefore, the total number of possible configurations is:

$$11 + 18 + 10 + 4 + 6 = 49$$

□

We can now give a small description of the Bronowski varieties having cubic fundamental surface.

Theorem 3.6.2. *Let X be a Bronowski threefold having fundamental surface of degree three. Suppose that the linear system \mathcal{X} defining the inverse of a general tangential projection of X has degree seven and that its base locus scheme has pure dimension one, that is, it has no embedded points. The fundamental surface V associated to this projection is a general projection of a scroll $S(1, 2)$ from an outer point.*

Then \mathcal{X} has multiplicity four in s , the double line of V , multiplicity two in a degree six curve C_6 and multiplicity one in r , the line in V skew to s . There is a pencil of cubics \mathcal{Q} containing the base locus of \mathcal{X} .

The threefold X has the following properties:

- (i) *It is an OADP variety;*
- (ii) *There is a cone F over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ with vertex a line L , such that X is the residual intersection of F with two quadrics containing a \mathbb{P}^4 of its ruling;*
- (iii) *The singularities of X have multiplicity two or three, its singular curves are lines and the triple points of X lie on L ;*

(iv) *The pencil \mathcal{Q} is mapped to a one-dimensional family of quartic surfaces \mathcal{Q}' in X with base locus L . This family is the intersection of X with the \mathbb{P}^4 's of the ruling of F .*

In Table 3.1, some configurations of the base locus of \mathcal{X} are presented, with the singularities of the corresponding OADP threefold X being described.

Proof. The fact that V is a general projection of $S(1, 2)$ follows from Proposition 3.1.1 and Proposition 3.5.4. The base locus of \mathcal{X} under the pure dimension hypothesis is given in Proposition 3.1.3, which also implies that \mathcal{X} is relatively complete. Then properties (i) and (ii) follow from Proposition 3.2.17. Property (iii) follows from Section 3.2.2, where the singularities of X are discussed. In Section 3.2.3, the family \mathcal{Q}' is studied, proving (iv). \square

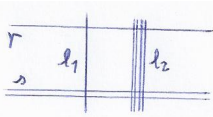
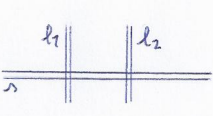
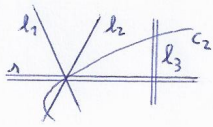
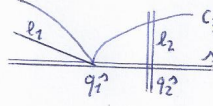
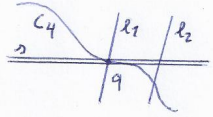
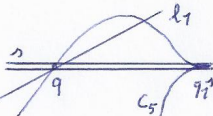
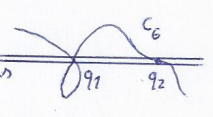
C_6	Figure of C_6 and s	Singularities of X
$r + \ell_1 + 4\ell_2$, with $\ell_1 \cap s \neq \ell_2 \cap s$ and $\ell_2 \cap s \neq q_1^s, q_2^s$		four double lines $R_2 \prec R_3 \prec R_4 \prec R_5$ and one double point
$2s + 2\ell_1 + 2\ell_2$		one triple line and two double lines
$C_2 + \ell_1 + \ell_2 + 2\ell_3$, as in Section 3.4		one double line, one triple point and one double point
$C_3 + \ell_1 + 2\ell_2$, with C_3 singular in q_1^s , $\ell_i \cap s = q_i^s$, for $i = 1, 2$		one double line R and two triple points not lying on R
$C_4 + \ell_1 + \ell_2$, with C_4 smooth, $\ell_1 \cap s = q$, $C_4 \cap s = 3q$ and C_4 has no other intersection with ℓ_1		two double points
$C_5 + \ell_1$, with C_5 cuspidal in q_1^s , $C_5 \cap s = 3q_1^s + q$ and C_5 intersects ℓ_1 in two points		two triple and one double points
C_6 , with a double point in q_1 and $C_6 \cap s = 2q_1 + 3q_2$		one triple point

Table 3.1: Example of Bronowski varieties having cubic fundamental surface.

Chapter 4

The quartic case

Finally, we will study the case in which the fundamental surface V has degree four. By [CMR], there are no smooth OADP threefolds fitting in this situation. Since we are assuming hypothesis (H), \mathcal{X} has degree nine. A further hypothesis will be made on the base locus of \mathcal{X} , see Proposition 4.1.4.

4.1 First Considerations

The map $\bar{\tau} : E \cong \mathbb{P}^2 \dashrightarrow V \subset \mathbb{P}^3$ is defined by a base point free non complete linear system of conics. Then V is a projection of the Veronese quartic surface from a disjoint line.

When computing self-intersections of curves in V , we'll consider its normalization, the Veronese surface.

The following proposition describes the three possibilities for these quartic surfaces. This description can be found in [Cof].

Proposition 4.1.1. *Let $V \subset \mathbb{P}^3$ be a projection of the Veronese quartic surface from a line disjoint to it. Then V is projectively equivalent to one of the following surfaces:*

- (i) $x_1^2x_2^2 - x_0x_1x_2x_3 + x_1^2x_3^2 + x_2^2x_3^2 = 0$
- (ii) $x_0^2x_2^2 - 2x_0x_1x_2^2 - x_0x_1x_3^2 + x_1^2x_2^2 + x_1^2x_3^2 - x_3^4 = 0$
- (iii) $4x_0^3x_1 - 13x_0^2x_1^2 + 14x_0x_1^3 - 5x_1^4 + 10x_0^2x_1x_2 - 22x_0x_1^2x_2 + 12x_1^3x_2 -$
 $- x_0^2x_2^2 + 10x_0x_1x_2^2 - 10x_1^2x_2^2 - 2x_0x_2^3 + 4x_1x_2^3 - x_2^4 - 8x_0^2x_3^2 + 20x_0x_1x_3^2 -$
 $- 12x_1^2x_3^2 - 12x_0x_2x_3^2 + 16x_1x_2x_3^2 - 4x_2^2x_3^2 - 4x_3^4 = 0$

The surface given by (i) is the general projection and is known as Steiner's Roman Surface. It has a triple point $p = (1 : 0 : 0 : 0)$ and three double lines:

$$\ell_1 : (x_2 = x_3 = 0) \quad ; \quad \ell_2 : (x_1 = x_3 = 0) \quad ; \quad \ell_3 : (x_1 = x_2 = 0)$$

which intersect in p . In the surface (ii), one of the three double lines is infinitely near to another line. In (iii), two lines are infinitely near.

The computations that follow will be done for the three cases, the difference being that the singular lines of V may be infinitely near.

The linear system $\Pi_{X,x}$ in $E \cong \mathbb{P}^2$ has no base points and maps a general line to a conic. There are three lines $\hat{\ell}_i \subset E$ that are mapped by $\bar{\tau}$ to the double lines ℓ_i of V . This gives, for each i , a double cover of $\ell_i \cong \mathbb{P}^1$, ramified in two points. Let $q_1^i, q_2^i \in \ell_i$ be the two branch points of this double cover.

Given a conic in E , it lies on $\Pi_{X,x}$ if and only if its two points of intersection with each line $\hat{\ell}_i$ are mapped to the same point in V . If the point is one of the ramification points, the conic must be tangent to $\hat{\ell}_i$.

We start with a small result on the preimages of $p \in V$ in E .

Lemma 4.1.2. *Consider the three surfaces given in Proposition 4.1.1 and the map $\bar{\tau} : E \dashrightarrow V$ defined by the linear system $\Pi_{X,x}$ in each case.*

If ℓ_1, ℓ_2, ℓ_3 are proper lines of V , then p has three preimages in E , namely $\hat{\ell}_i \cap \hat{\ell}_j$.

If, for instance, ℓ_2 is infinitely near to ℓ_1 , then p is a branch point of the double cover of ℓ_3 , whereas it has two preimages via $\bar{\tau}$ in $\hat{\ell}_1$.

On the other hand, if $\ell_1 \prec \ell_2 \prec \ell_3$, then p has only one preimage and it is a branch point in ℓ_1 .

Proof. Let u_0, u_1, u_2 be projective coordinates in $E \cong \mathbb{P}^2$. The first assertion is clear, since in this case the lines $\hat{\ell}_i \subset E$ are not infinitely near.

To prove the second assertion, consider the equation given in Proposition 4.1.1, item (ii). Then the two proper double lines of V are:

$$\ell_3 : x_2 = x_3 = 0 \quad ; \quad \ell_1 : x_0 - x_1 = x_3 = 0$$

and $p = (1 : 1 : 0 : 0)$.

Note that setting $x_0 = x_1$ in the equation of V gives $x_3^4 = 0$, that is, V intersects the plane given by $(x_0 - x_1 = 0)$ in $4\ell_1$. Therefore, blowing up ℓ_1 , V has a second double line ℓ_2 infinitely near to it given by this plane. Then this notation agrees with the hypothesis that ℓ_2 is infinitely near to ℓ_1 .

The map $\bar{\tau}$ can be given by:

$$(u_0 : u_1 : u_2) \mapsto (u_0^2 - u_1^2 + u_2^2 : u_2^2 - u_1^2 : u_1 u_2 : u_0 u_1)$$

Then the preimages of p are $(0 : 0 : 1)$ and $(0 : 1 : 0)$. Moreover:

$$\hat{\ell}_3 : u_1 = 0 \quad ; \quad \hat{\ell}_1 : u_0 = 0$$

so both preimages lie on $\hat{\ell}_1$.

The double cover of ℓ_3 is given by:

$$(u_0 : u_2) \mapsto (x_0 : x_1) = (u_0^2 + u_2^2 : u_2^2)$$

and p has only one preimage, namely $(0 : 1)$. This proves the second part.

For the last assertion, consider the variety given by (iii) in Proposition 4.1.1. The double line of V is:

$$\ell_1 : x_0 - x_1 + x_2 = x_3 = 0$$

and $p = (2 : 1 : -1 : 0)$. Then $\bar{\tau}$ is given by:

$$(u_0 : u_1 : u_2) \mapsto (u_0^2 + 2u_1^2 + u_2^2 : 2u_1^2 + u_2^2 : u_2^2 + 2u_0 u_2 : u_1 u_2 + u_0 u_1)$$

It follows then that the only preimage of p is $(1 : 0 : -1)$. In particular, p is a branch point of the double cover of ℓ_1 . □

For $i \in \{1, 2, 3\}$, let Γ_i be the plane containing the lines ℓ_j , with $j \neq i$. Then:

$$V \cap \Gamma_i = \ell_j + \ell_k$$

for i, j, k distinct. If, for example, ℓ_2 is infinitely near to ℓ_1 , then the plane Γ_1 does not exist and ℓ_2 is defined in the exceptional divisor E_{ℓ_1} of the blow up in ℓ_1 by the intersection with Γ_3 . If ℓ_2 is infinitely near to ℓ_1 and ℓ_3 is infinitely near to ℓ_2 , then there is only the plane Γ_3 .

Consider the blow up at p . Then V_p is a degree three curve with three double points, some of them possibly infinitely near. Therefore it is the union of three lines, which may include infinitely near lines.

If V is the Steiner's Roman Surface, then:

$$V_p = t_1 + t_2 + t_3$$

with $t_i = \Gamma_i \cap E_p$.

If the line ℓ_2 is infinitely near to ℓ_1 , then V_p has two proper and one infinitely near double points. This gives:

$$V_p = 2t_2 + t_3$$

After that, the blow up at ℓ_1 gives $V_{\ell_1} \equiv (1, 2)$, since V intersects a plane through ℓ_1 in a conic through p . Then:

$$V_{\ell_1} = f^p + 2\ell_2 \equiv (1, 2) \quad (4.1)$$

where $f^p = E_p \cap E_{\ell_1}$ and $\ell_2 = \Gamma_3 \cap E_{\ell_1}$. In particular, t_2 is not a double curve of V , that is, blowing up t_2 gives $V_{t_2} = E_p \cap E_{t_2}$. To keep the notation of the general case, set $t_1 = V_{t_2}$.

If we are in case (iii) of Proposition 4.1.1, say $\ell_1 \prec \ell_2 \prec \ell_3$, then:

$$V_p = 3t_3$$

with $t_3 = (\Gamma_3)_p$. Blowing up ℓ_1 gives:

$$V_{\ell_1} = f_1^p + 2\ell_2 \equiv (1, 2)$$

with $f_1^p = E_p \cap E_{\ell_1}$ and $\ell_2 = \Gamma_3 \cap E_{\ell_1}$. Since $(\ell_2)^2 = 0$ in both Γ_3 (which was blown up in p) and E_{ℓ_1} , the blow up at ℓ_2 gives $E_{\ell_2} \cong \mathbb{F}_0$. The intersections of V with Γ_3 and E_{ℓ_1} give $V_{\ell_2} \equiv (1, 2)$. Then:

$$V_{\ell_2} = f_2^p + 2\ell_3 \equiv (1, 2) \quad (4.2)$$

where $f_2^p = E_p \cap E_{\ell_2}$. Since $V_p = 3t_3$, the blow up at t_3 gives:

$$V_{t_3} = t_1 = E_p \cap E_{t_3}$$

and the blow up at t_1 gives:

$$V_{t_1} = t_2 = E_p \cap E_{t_1}$$

See Figure 4.1.

In conclusion, in the three cases we have:

$$V_p = t_1 + t_2 + t_3 \quad (4.3)$$

where some of these lines can be infinitely near.

These considerations explain the tangent cone of V in p . Next we study the tangent cone of V in other singular points. These results are very similar to the case in which V has degree three, given in chapter 3.

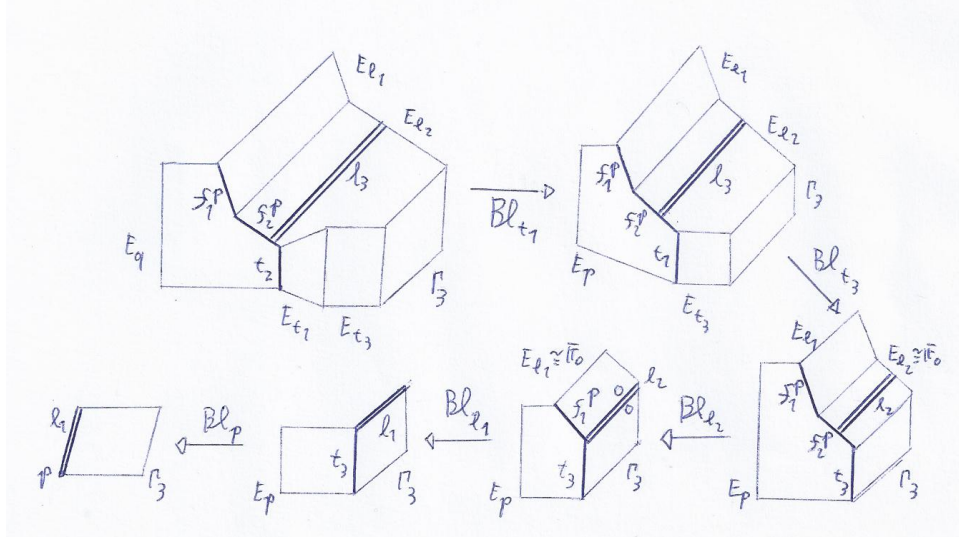


Figure 4.1: Blow ups of V when $l_1 < l_2 < l_3$

Lemma 4.1.3. *Suppose V does not have a double line infinitely near to l_i . Then the tangent cone of V in a point $q \in l_i \setminus \{p, q_1^i, q_2^i\}$ is a pair of distinct planes containing l_i . If $q = q_j^i$, the tangent cone of V in q is a double plane containing l_i . In both cases, V intersects one of these planes in $2l_i$ and a conic through p and q .*

On the other hand, if for instance l_2 is infinitely near to l_1 , then the tangent cone of V in a point $q \in l_1$ distinct from p is $2\Gamma_3$.

Proof. In the first case, set $i = 1$. Since l_1 is a double line of V , the tangent cone of V in a point of l_1 different from p is a pair of planes containing l_1 . These planes may coincide or not.

Let q be a point in l_1 different from p and let Π be a plane in the tangent cone of V in q . Then $\Pi \cap V$ has multiplicity three in p and q , so it is the union of $2l_1$ and a conic C through p and q . Note that $2\hat{l}_1 \notin \Pi_{X,x}$, since the hypothesis that V does not have a double line infinitely near to l_1 implies that $\hat{l}_1 \cap \hat{l}_j$ is not a ramification point of \hat{l}_j , for $j \in \{2, 3\}$. Therefore $4l_1$ is not a plane section of V , whence $C \neq 2l_1$.

The point q has one or two preimages in \hat{l}_1 , and p has two or three preimages (depending on V), namely $\hat{l}_i \cap \hat{l}_j$. The strict transform of C cannot be \hat{l}_1 , so it is a line through one of the points $\bar{\tau}^{-1}(q) \in \hat{l}_1$ and the preimage of p not lying on \hat{l}_1 .

If q is not q_1^1 or q_2^1 , there are two such lines, so there are two distinct

conics C . Therefore the two planes in the tangent cone of V in q are distinct. If q is q_1^1 or q_2^1 , there is only one conic C and the tangent cone of V in q is a double plane. This proves the first part.

Now suppose ℓ_2 is infinitely near to ℓ_1 . The blow up at p and ℓ_1 gives:

$$V_{\ell_1} = f^p + 2\ell_2$$

where $\ell_2 = (\Gamma_3)_{\ell_1}$ and $f^p = E_{\ell_1} \cap E_p$, as explained above. Since $q \neq p$, the tangent cone of V in q is $2\Gamma_3$. □

The base locus of \mathcal{X} is now described:

Proposition 4.1.4. *Let ℓ_1, ℓ_2 and ℓ_3 be the double lines of V and let p be its triple point. Suppose that the base locus of \mathcal{X} has dimension one, except for p . Then \mathcal{X} is the linear system of surfaces with degree nine having:*

- *multiplicity six in p*
- *multiplicity four along each ℓ_i*
- *multiplicity two along C_6 ,*

where C_6 is a degree six curve in V , image via $\bar{\tau}$ of a cubic in E . In particular, C_6 cuts each ℓ_i in three points, supposing it does not contain any of these lines.

Moreover, X has degree nine and p is mapped to a point x_p of multiplicity four in X . If C_6 does not contain p , the tangent cone of X in x_p is a cone over a Veronese quartic surface.

Proof. Let d_i be the multiplicity of \mathcal{X} in ℓ_i . Since \mathcal{X}'' is the linear system of planes, $d_i \leq 4$. And since \mathcal{X}' desingularizes V to E , $d_i - 2 \geq 1$. Therefore $d_i \in \{3, 4\}$

The linear system \mathcal{X}' has multiplicity $d_i - 2$ along ℓ_i . Its restriction to V defines the inverse of $\bar{\tau}$. Then:

$$\mathcal{X}' \cap V = 2(d_1 - 2)\ell_1 + 2(d_2 - 2)\ell_2 + 2(d_3 - 2)\ell_3 + \{\text{conics}\} + C$$

for a fixed curve C . Since $\deg \mathcal{X}' = 5$, this implies:

$$\deg C = 20 - 2(d_1 - 2) - 2(d_2 - 2) - 2(d_3 - 2) - 2 \quad (4.4)$$

But \mathcal{X} has multiplicity two along C . So its intersection with V is:

$$\mathcal{X} \cap V = 2d_1\ell_1 + 2d_2\ell_2 + 2d_3\ell_3 + 2C + D$$

for a certain curve D . This implies:

$$36 \geq 2d_1 + 2d_2 + 2d_3 + 2 \deg C$$

Plugging (4.4) in the above inequality, gives:

$$d_1 + d_2 + d_3 \geq 12$$

Therefore $d_i = 4$ for $i = 1, 2, 3$. In particular there is no curve D and $C = C_6$ has degree six. This proves the assertion on the base locus.

Therefore the intersection of \mathcal{X} with V is:

$$\mathcal{X} \cap V = 8\ell_1 + 8\ell_2 + 8\ell_3 + 2C_6$$

A general plane is cut by \mathcal{X} in degree nine curves with three points of multiplicity four and six double points. It is mapped to a general tangent hyperplane section of X . Then X has degree $81 - 3 \cdot 4^2 - 6 \cdot 2^2 = 9$.

To find the multiplicity of \mathcal{X} in p , consider the blow up at this point. Then \mathcal{X}_p has multiplicity four in three non collinear points, so $\text{mult}_p \mathcal{X} \geq 6$. But since \mathcal{X}'' has no base points and p is a triple point of V , $\text{mult}_p \mathcal{X} \leq 6$ and equality holds.

Since \mathcal{X}_p has degree six and three points of multiplicity four, it is the union of three fixed double lines. Some of these lines are infinitely near if V is not the Steiner's Roman Surface. Using the notation of (4.3), we have:

$$\mathcal{X}_p = 2t_1 + 2t_2 + 2t_3 \tag{4.5}$$

Therefore \mathcal{X}_p is a fixed curve and p is mapped to a point x_p in X . Let Ω be a general plane through p . Then Ω_p intersects \mathcal{X}_p in three (possibly infinitely near) fixed double points. After blowing up these points, Ω_p has no intersection with \mathcal{X} and $(\Omega_p)^2 = -4$ in Ω . Then x_p is a point of multiplicity four in the hyperplane section $\sigma(\Omega)$ of X . By Lemma 1.3.3, X has multiplicity four in x_p .

Finally, suppose $p \notin C_6$. Let $\hat{\mathcal{X}}$ be the linear system of surfaces in \mathcal{X} having multiplicity seven in p . Then, the blow up in p gives:

$$\hat{\mathcal{X}}_p = t_1 + t_2 + t_3 + \{\text{quartic curves}\}$$

The moving part has three double points, namely $t_i \cap t_j = \ell_k \cap E_p$, with i, j, k distinct. If V is not the Steiner's Roman surface, some of these points are infinitely near. Since $p \notin C_6$, it has no other base points. A standard quadratic Cremona transformation maps the moving part of $\hat{\mathcal{X}}$ to a linear

system of conics having no base points. Therefore, by Lemma 1.1.4 the projectivization of the tangent cone of X in x_p contains a Veronese surface. Since X has multiplicity four in x_p , the result follows. \square

Note that the possible irreducible components of C_6 are conics, quartics or the double lines ℓ_i . Indeed, C_6 is the image via $\bar{\tau}$ of a cubic in E . But $\bar{\sigma}$ maps any curve of degree d to a curve of degree $2d$, except for the three lines $\hat{\ell}_i$. This proves the assertion.

4.2 Images of ℓ_i

In this section we study the image of the lines ℓ_i via σ . We'll keep the hypothesis of Proposition 4.1.4, that is, \mathcal{X} has degree nine and pure dimension one except for p . We will first consider the case in which C_6 does not contain any of the lines ℓ_i , and then the other cases.

Proposition 4.2.1. *Suppose C_6 does not contain any of the lines of V and fix $i \in \{1, 2, 3\}$. If V has no double line infinitely near to ℓ_i , then ℓ_i is mapped by σ to a weak Del Pezzo surface ${}^iD_4^x$ of degree four through x having a double point in x_p . If there is a double line infinitely near to ℓ_i , then ℓ_i is mapped to a double line R_i of X not containing x_p .*

Proof. Set $\ell = \ell_i$. Suppose first that ℓ is a proper double line of V with no other double line infinitely near to it. Let Ω be a general plane containing ℓ . Then $\mathcal{X} \cap \Omega$ consists of 4ℓ and moving quintic curves, which intersect ℓ in p with multiplicity two and in three moving points.

After the blow up at p , the normal bundle of ℓ is:

$$N_\ell = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

Blowing up ℓ gives $\mathcal{X}_\ell \equiv (3, 4)$. Since there is no double line of V infinitely near to ℓ , \mathcal{X}_ℓ has no fixed components. It has two double points in $f_\ell^p = E_p \cap E_\ell$, given by the intersections with two of the lines t_i (see (4.3)). One of these points is infinitely near if V is not the Steiner's Roman surface. It also has three double points in the intersections with C_6 , which does not contain ℓ . The curve of type $(1, 2)$ through these five base points is V_ℓ and is mapped to x .

By Lemma 1.1.1, \mathcal{X}_ℓ is birationally equivalent to degree seven curves in \mathbb{P}^2 with one point of multiplicity four, one triple point and five double points. The fiber f_ℓ^p is mapped to a line through the triple point and two of

the double points. After two standard quadratic Cremona transformations, the degree seven curves are mapped to cubics with five base points. Three of these points lie on a line, the image of f_ℓ^p . Since f_ℓ^p is mapped to x_p , E_ℓ is mapped to a weak Del Pezzo quartic surface through x with a double point in x_p .

Now suppose ℓ is a double line of V infinitely near to ℓ' , which can be proper or infinitely near to another line ℓ'' . Blowing up p (and ℓ'' , if it is the case) and ℓ' , by (4.1) and (4.2) we have that $\ell \equiv (0, 1)$ in $E_{\ell'}$. As remarked above, $\mathcal{X}_{\ell'} \equiv (3, 4)$ in both cases. Moreover, it has two double points in $f_{\ell'}^p = E_p \cap E_{\ell'}$, due to (4.3) and (4.5). Therefore:

$$\mathcal{X}_{\ell'} = 4\ell + 2f_{\ell'}^p + \{\text{moving fibers}\} \equiv (0, 4) + (2, 0) + (1, 0) \quad (4.6)$$

Then ℓ' is mapped to a line R' .

If ℓ' is a proper line of V (and V is of type (ii) in Proposition 4.1.1), then $\ell = \Gamma \cap E_{\ell'}$, with $\Gamma \cong \mathbb{F}_1$ being the strict transform of the plane spanned by ℓ' and ℓ .

If ℓ' is infinitely near to ℓ'' , then $\ell = Q \cap E_{\ell'}$, with Q being the strict transform of a quadric containing the three double lines of V . This implies that the original quadric Q is singular in p , since Q_p contains three non collinear points. After the blow up at p we get, in $Q \cong \mathbb{F}_2$, $\ell'' \equiv f_2$ and $Q_p \equiv e_2$. Then:

$$V \cap Q = 6\ell'' + F \equiv e_2 + 8f_2$$

where $F \equiv e_2 + 2f_2$ is a plane section of Q . Blowing up ℓ'' and ℓ' does not affect Q , and gives $\ell \equiv f_2$ in Q . Analogously, the intersection of \mathcal{X} with Q is:

$$\mathcal{X} \cap Q = 12\ell'' + G \equiv 3e_2 + 18f_2$$

where $G \equiv 3e_2 + 6f_2$ is a cubic section of Q .

Therefore, in both cases ℓ has normal bundle:

$$N_\ell = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

and blowing up ℓ gives $V_\ell \equiv (1, 2)$ and $\mathcal{X}_\ell \equiv (3, 4)$. \mathcal{X}_ℓ has two double points in $E_p \cap E_\ell$, and three double points in $C_6 \cap E_\ell$, giving the same linear system as before. Then E_ℓ is mapped to a weak Del Pezzo surface D_4^x .

Next we prove that R' does not contain x_p . First note that $R' \subset D_4^x$, since $(E_\ell)_{\ell'}$ is a curve of type $(0, 1)$ in $E_{\ell'}$, so it is mapped to R' . Now, one of the two double points in $E_p \cap E_\ell$ is $f_{\ell'}^p \cap E_\ell \subset E_{\ell'} \cap E_\ell$. The other double point is infinitely near to it if ℓ' is not a proper line of V .

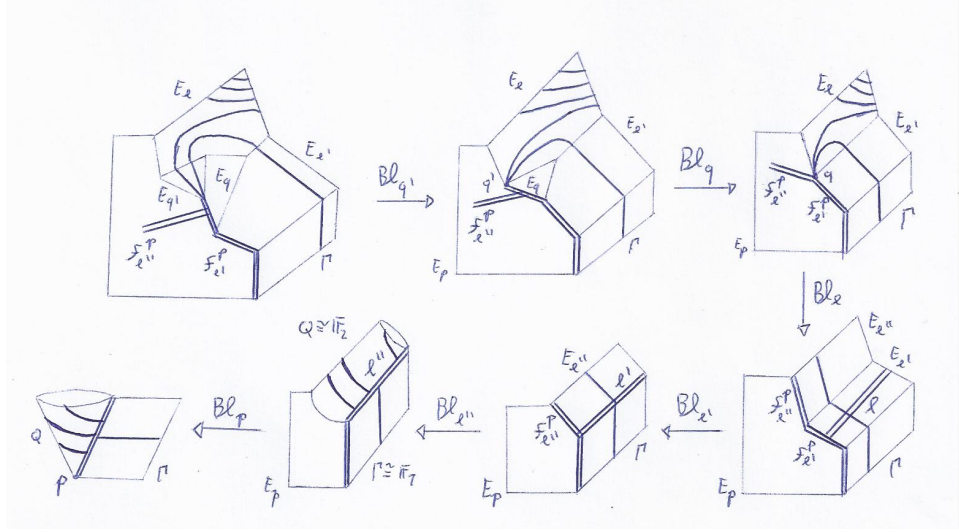


Figure 4.2: Blow ups when $\ell'' \prec \ell' \prec \ell$

Now note that $E_p \cap E_\ell$ and $E_{\ell'} \cap E_\ell$ are transversal and consider the blow up of the point $q = f_{\ell'}^p \cap E_\ell$, which is the intersection of these two curves. If ℓ' is a proper line of V , \mathcal{X}_ℓ intersects the exceptional divisor in two moving points, mapping it to a conic. If ℓ' is infinitely near to ℓ'' , \mathcal{X}_ℓ intersects the exceptional divisor in another double point q' , lying on $E_\ell \cap E_p$. Blowing up this point, \mathcal{X}_ℓ intersects the new exceptional divisor in two moving points. So in both cases, the last exceptional divisor is mapped to a conic. One point of this conic is x_p , image of $E_\ell \cap E_p$. And a distinct point of it lies on the image of $E_{\ell'} \cap E_\ell$, namely R' . Therefore $x_p \notin R'$, since no other curve of E_ℓ intersecting E_p and $E_{\ell'}$ is contracted.

In Figure 4.2 is a sketch of these blow ups when $\ell'' \prec \ell' \prec \ell$.

To prove that ℓ' is mapped to a double line of X , suppose it is a proper line of V and let q be a general point of ℓ' and Ω be a general plane through q . Then $\mathcal{X} \cap \Omega$ has multiplicity four in q . Blowing up q , by (4.6) $\mathcal{X} \cap \Omega$ has a fixed point of multiplicity four in Ω_q , corresponding to ℓ . Blowing up this point, Ω_q has self-intersection -2 in Ω and is not cut by $\mathcal{X} \cap \Omega$. Then it is mapped to a double point in the image of Ω , a hyperplane section of X . By Lemma 1.3.3, it is a double point of X , which implies that the image of ℓ' is a double line.

Clearly the same reasoning applies if $\ell'' \prec \ell' \prec \ell$, giving a double line R'' and a second double line R' infinitely near to it.

□

A similar result holds when C_6 contains double lines of V . We'll study a simple case to illustrate this situation.

First note that $\ell_i \subset C_6$ means that the cubic curve $\bar{\sigma}(C_6) \subset E$ contains the line $\hat{\ell}_i$. This implies that $C_6 = 2\ell_i + C_4$. Since \mathcal{X} has multiplicity four in ℓ_i , this means that \mathcal{X} has a double curve infinitely near to this line. This double curve is determined by V , since $C_6 \subset V$. In other words, blowing up p and then $\ell = \ell_i$, the linear system \mathcal{X} has multiplicity two along the curve V_ℓ . The reason we blow up p first is to avoid fixed components, since $\text{mult}_p \mathcal{X} > \text{mult}_\ell \mathcal{X}$.

From these considerations, it follows that if, for example, ℓ_3 lies infinitely near to ℓ_1 , then $\ell_1 \subset C_6$ implies $\ell_3 \subset C_6$. So we write $C_6 = 2\ell_3 + C_4$.

Below is a result similar to Proposition 4.2.1, supposing $\ell_1 \subset C_6$.

Lemma 4.2.2. *Suppose that $C_6 = 2\ell_1 + C_4$ and suppose V is the Steiner's Roman Surface. Then ℓ_1 is mapped to a triple line L_1 of X through x_p , and the double curve of \mathcal{X} infinitely near to ℓ_1 is mapped to a projection of a quartic Veronese surface through x having multiplicity two along L_1 and spanning a \mathbb{P}^4 .*

Proof. Set $\ell = \ell_1$. The blow ups that will now be done are represented in Figure 4.3.

Start blowing up p . Set $t_i = (\Gamma_i)_p$. Then:

$$\mathcal{X}_p = 2t_1 + 2t_2 + 2t_3 = 2V_p$$

The line ℓ is the complete intersection of Γ_2 and Γ_3 . After the blow up at p , we get $\ell^2 = 0$ in both surfaces. Blow up ℓ . We want to study the image of $E_\ell \cong \mathbb{F}_0$ via σ . As remarked above, since $C_6 = 2\ell + C_4$, \mathcal{X}_ℓ has multiplicity two along V_ℓ .

A general plane through ℓ is intersected by V in 2ℓ and a conic through p . Therefore $V_\ell \equiv (1, 2)$, a rational curve. On the other hand, a plane through ℓ is cut by \mathcal{X} in 4ℓ and quintic curves having a double point in p . Then $\mathcal{X}_\ell \equiv (3, 4)$. This implies:

$$\mathcal{X}_\ell = 2u + \{\text{fibers}\} \equiv (3, 4)$$

where $u = V_\ell$ and the moving part consists of fibers in E_ℓ . Therefore E_ℓ is mapped to a line L_1 . Since $(E_p)_\ell \equiv (1, 0)$, L_1 contains x_p . Note that the moving part of \mathcal{X}_ℓ cuts u in a degree two non complete linear series.

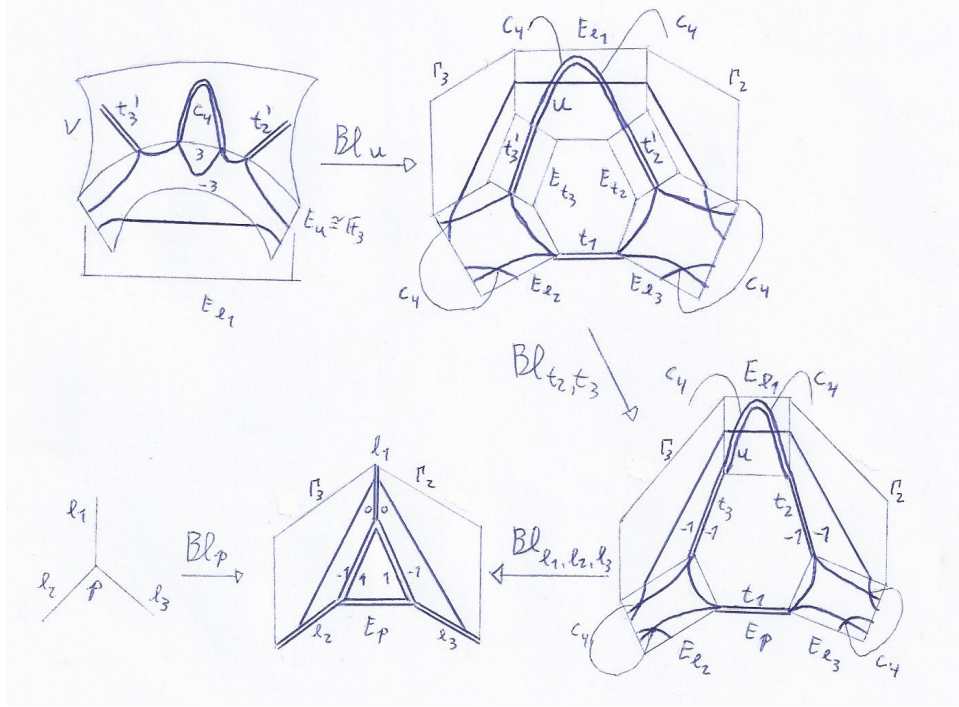


Figure 4.3: Blow ups when $C_6 = 2\ell_1 + C_4$

Before proceeding to the investigation of the image of u , other blow ups will be made in order to bring to light the existence of base curves of \mathcal{X} infinitely near to t_2 and t_3 . First consider the blow up of the lines ℓ_2 and ℓ_3 , since \mathcal{X} has multiplicity four along them.

The curve t_2 is the complete intersection of Γ_2 and E_p . After the blow ups along the three lines ℓ_i , t_2 has self-intersection (-1) in both surfaces. The same holds for t_3 . Then, for $i = 2, 3$ the normal bundle of t_i is:

$$N_{t_i} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

Set $t = t_2$. Remember that the moving part of $\mathcal{X} \cap \Gamma_2$ does not intersect t . Then, blowing up t , we get $\mathcal{X}_t \equiv (0, 2)$. These curves have multiplicity two in the point u_t , therefore $\mathcal{X}_t = 2t'_2$, where t'_2 is the fiber containing u_t . Since t_2 did not intersect t_3 , this blow up did not affect t_3 . So the same reasoning can be made, giving $\mathcal{X}_{t_3} = 2t'_3$.

The curves t'_2 and t'_3 are actually double curves of \mathcal{X} (infinitely near to t_2 and t_3). To prove this assertion, let Ω be a general plane through p and

denote intersections with Ω with an index (for instance, $\mathcal{X} \cap \Omega = \mathcal{X}_\Omega$). We have to repeat the blow ups made above and prove that \mathcal{X}_Ω has multiplicity two in $(t'_2)_\Omega$ and $(t'_3)_\Omega$. Before the blow ups, \mathcal{X}_Ω has multiplicity six in p and multiplicity two in four points $(C_4)_\Omega$. After the blow up of p , \mathcal{X}_Ω has three double points on the exceptional divisor, namely $(t_i)_\Omega$, with $i = 1, 2, 3$. Blowing up $\ell_1, \ell_2, \ell_3, t_2$ and t_3 , we have $t'_2 \subset E_{t_2}$ and $t'_3 \subset E_{t_3}$. Then the degree of the image of Ω via \mathcal{X} is:

$$d = 9^2 - 6^2 - 4 \cdot 2^2 - 3 \cdot 2^2 - 2 \cdot \delta = 17 - 2\delta$$

where δ corresponds to $(t'_j)_\Omega$, for $j = 2, 3$. If it is a double point of \mathcal{X}_Ω , then $\delta = 4$. If it is a base point with a second infinitely near base point, then $\delta = 2$. But the image of Ω is a tangent hyperplane section of X , giving $d = 9$ and $\delta = 4$. This proves the assertion.

Now let's investigate the normal bundle of $u = E_\ell \cap V$. As already noted, u is rational. In E_ℓ , two points of u were blown up, so $u^2 = 4 - 2 = 2$. In V , u is the line ℓ blown up at p , since all the curves that were blown up lied in V . To compute ℓ^2 , we look at the curve $\bar{\ell}$ in the normalization \bar{V} of V that is mapped to ℓ . It is a conic and two of its points are mapped to p . The self-intersection of a conic in the Veronese variety \bar{V} is 1. Then $u^2 = 1 - 2 = -1$. Therefore the normal bundle of u is:

$$N_u = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

Blowing up u gives $E_u \cong \mathbb{F}_3$, $V_u \equiv e_3 + 3f_3$ and $(E_\ell)_u \equiv e_3$. \mathcal{X}_u intersects $(E_\ell)_u$ in two points moving in a non complete linear series and intersects V_u in four fixed double points. These points are $(t'_2)_u, (t'_3)_u$ and two points in $(C_4)_u$. Therefore:

$$\mathcal{X}_u \equiv 2e_3 + 8f_3$$

having four double base points. By Lemma 1.1.1, \mathcal{X}_u can be birationally mapped to a linear system in \mathbb{P}^2 of degree eight curves with four double points, a point of multiplicity six and two double points infinitely near to it. After applying three standard quadratic maps in \mathbb{P}^2 , these curves are mapped to conics with no base points. Since \mathcal{X}_u is not complete, this is a non complete linear system. Therefore the image of E_u is a quartic surface, a projection of a Veronese surface. The curve $(E_\ell)_u$ is mapped to a double line of it, namely L_1 , and V_u is contracted to x . Since it has no other double lines (proper or infinitely near), this surface spans a \mathbb{P}^4 .

We are left to prove that L_1 is a triple line of X . Note that a point in ℓ_1 is blown up and mapped back to a point in L_1 . So let q be a general

point in ℓ_1 , let x_q be its image in X and let Ω be a general plane through q . Blowing up p and $\ell = \ell_1$ as above, we see that Ω_ℓ is the fiber over the point q , intersecting u in two points.

Therefore $\mathcal{X} \cap \Omega$ has multiplicity four in q and two double points infinitely near to it. After the blow up at these points, the self-intersection of the exceptional divisor of q in Ω is -3 , having no intersection with $\mathcal{X} \cap \Omega$. Then Ω is mapped to a tangent section of X having multiplicity three in x_q . By Lemma 1.3.3, X has multiplicity three in x_q and L_1 is a triple line. \square

If C_6 contains other lines, the threefold X will have other triple lines. It may also happen that $C_6 = 4\ell_1 + C_2$ or $C_6 = 6\ell_1$, in which cases X has singularities infinitely near to L_1 . Similar results also hold when V is not the Steiner's Roman Surface. We will not dwell on this here.

4.3 Quartic surfaces in X

A consequence of Proposition 4.2.1 and Lemma 4.2.2 is that there is at least one quartic surface through x in X having a double point in x_p . If C_6 does not contain lines, then it is a weak Del Pezzo surface. Otherwise it can be a surface with a double line. In both cases, this surface spans a \mathbb{P}^4 .

Since x is a general point of X , these surfaces cover X . The following Lemma explains their images in \mathbb{P}^3 .

Lemma 4.3.1. *Fix $i \in \{1, 2, 3\}$ and suppose V does not have a double line infinitely near to ℓ_i . Let \mathcal{Q}_i be the linear system of quartic surfaces in \mathbb{P}^3 containing C_6 and ℓ_i and having multiplicity two in ℓ_j , for $j \neq i$. Then \mathcal{Q}_i is a pencil and it is mapped to a family \mathcal{Q}'_i of quartic surfaces covering X . The surface in \mathcal{Q}'_i corresponding to $V \in \mathcal{Q}_i$ is the image of ℓ_i .*

Proof. The last assertion follows from the fact that V is the surface in \mathcal{Q}_i with an extra multiplicity along ℓ_i and from the fact that V is contracted to x .

To simplify the notation, set $i = 1$ and $\mathcal{Q} = \mathcal{Q}_1$. Let us first show that there is a surface different from V in \mathcal{Q} . We'll obtain such surface as the union of a cubic S_3 and a plane. By Lemma 3.2.7, there is at least a pencil of cubic surfaces in \mathbb{P}^3 containing C_6 . This holds in the general case, where C_6 is an elliptic curve, so it must also hold in the degenerated cases.

If $p \notin C_6$, let S_3 be the cubic containing C_6 and p . By Proposition 4.1.4, C_6 cuts each of the lines ℓ_1, ℓ_2, ℓ_3 in three points, so S_3 intersects these lines in four points. Then S_3 contains C_6, ℓ_1, ℓ_2 and ℓ_3 .

If C_6 is smooth in p , then it is the image via $\bar{\tau}$ of a cubic in E through one of the three preimages of p , say $\hat{\ell}_1 \cap \hat{\ell}_2$. This cubic intersects $\hat{\ell}_1$ and $\hat{\ell}_2$ in two other points each, and intersects $\hat{\ell}_3$ in three points. This implies that every cubic surface containing C_6 intersects ℓ_3 in p and in other three points, so it contains ℓ_3 . In particular, these cubic surfaces have fixed tangent plane in p . Therefore there is a cubic S_3 containing C_6 and having a double point in p . This cubic contains the three lines ℓ_1, ℓ_2, ℓ_3 .

Similar considerations prove that if p is a singular point of C_6 , then there is a cubic surface S_3 containing C_6 , ℓ_1 , ℓ_2 and ℓ_3 . This can be achieved by doing a case by case analysis in the types of singularities of C_6 and its intersections with these lines.

Now, consider the union of S_3 and the plane Γ_1 , which exists since V does not have a double line infinitely near to ℓ_1 . This is a surface in \mathcal{Q} different from V .

To prove that \mathcal{Q} is a pencil, let S be a quartic in \mathcal{Q} different from V . Then:

$$S \cap V = 2\ell_1 + 4\ell_2 + 4\ell_3 + C_6 \quad (4.7)$$

that is, the surfaces in \mathcal{Q} have fixed intersection with V . Repeating the argument used in Section 3.2.3, the exact sequence:

$$0 \rightarrow H^0(\mathbb{P}^3, \mathcal{Q} - V) \rightarrow H^0(\mathbb{P}^3, \mathcal{Q}) \rightarrow H^0(\mathbb{P}^3, \mathcal{Q}|_V) \rightarrow 0$$

shows that \mathcal{Q} is a pencil.

Next we compute the image of S via σ . Intersecting with \mathcal{X} gives:

$$\mathcal{X} \cap S = 4\ell_1 + 8\ell_2 + 8\ell_3 + 2C_6 + F_4$$

where F_4 is the moving part. However, note that the fixed part is the intersection of $2V$ with S , so F_4 is a base point free plane section of S . Then the image of S is a quartic surface in X .

The surfaces in \mathcal{Q}' clearly cover X , since \mathcal{Q} covers \mathbb{P}^3 . □

Two remarks have to be made on the linear systems \mathcal{Q}_i . Note that if, for instance, ℓ_2 is infinitely near to ℓ_1 (i.e., V is not the Steiner's Roman Surface), then the surfaces in \mathcal{Q}_1 have multiplicity two along ℓ_2, ℓ_3 and multiplicity one along ℓ_1 . Applying Enriques's unloading principle (cf. [EC], IV.17), we get surfaces having multiplicity two along ℓ_1 and ℓ_3 and containing ℓ_2 . Therefore \mathcal{Q}_1 coincides with \mathcal{Q}_2 . To avoid confusion, we will not consider \mathcal{Q}_1 in this case. This explains the hypothesis made in the Lemma.

A second remark is that C_6 may contain a double line of V , say ℓ_1 . In this case, the surfaces of \mathcal{Q}_i (for $i = 1, 2, 3$) have in common a curve u infinitely near to ℓ_1 , determined by V , in order to have (4.7) (and corresponding equations for $i = 2, 3$). This is the same double curve of \mathcal{X} , of type $(1, 2)$ in E_{ℓ_1} , as it is explained in the proof of Lemma 4.2.2.

This implies that any surface containing ℓ_1 and u actually has multiplicity two along ℓ_1 . Therefore \mathcal{Q}_1 is, in the case where $C_6 = 2\ell_1 + C_4$, the linear system of surfaces having multiplicity two along ℓ_1, ℓ_2, ℓ_3 and containing C_4 . This gives, for $S \in \mathcal{Q}_1$ different from V :

$$S \cap V = 4\ell_1 + 4\ell_2 + 4\ell_3 + C_4$$

which agrees with (4.7). Note that in this case, u is not a base curve of \mathcal{Q}_1 , since these surfaces already have multiplicity two along ℓ_1 .

On the other hand, \mathcal{Q}_2 is the linear system having multiplicity two along ℓ_1 and ℓ_3 and containing C_4, u and ℓ_2 . And the linear system \mathcal{Q}_3 is defined in the obvious way. In all these cases, the proof that \mathcal{Q}_i is a pencil follows from the same argument given in Lemma 4.3.1.

The next result gives more details on the families \mathcal{Q}'_i .

Lemma 4.3.2. *The base locus of \mathcal{Q}'_i is a line L_i through x_p . There are possibly other base curves infinitely near to L_i , in case $\ell_i \subset C_6$. Moreover, if $\ell_i \not\subset C_6$, then the general surface in \mathcal{Q}'_i is a quartic weak Del Pezzo surface having a double point in $x_p \in L_i$. If $\ell_i \subset C_6$, it is a quartic surface having multiplicity two along L_i .*

Proof. By the definition of \mathcal{Q}_i , V does not have a double line infinitely near to ℓ_i . By Lemma 4.2.1, if $\ell_i \not\subset C_6$ then ℓ_i is mapped to a quartic weak Del Pezzo surface through x having a double point in $x_p \in L_i$. This is the surface of \mathcal{Q}'_i through x . Since x is a general point of X , this is a general surface of \mathcal{Q}'_i . If $\ell_i \subset C_6$, the result follows from Lemma 4.2.2.

To prove the assertion on the base locus, suppose $i = 1$ and set $\mathcal{Q} = \mathcal{Q}_i$. Assume first that $\ell_1 \not\subset C_6$. Equation (4.7) gives us the base locus of \mathcal{Q} , consisting of base curves of \mathcal{X} , and a curve infinitely near to ℓ_1 , since a general surface of \mathcal{Q} is smooth along ℓ_1 .

The base locus of \mathcal{Q}' arises either from curves in the base locus of \mathcal{Q} not lying on the base locus of \mathcal{X} or from curves in \mathcal{Q} that are contracted by σ . Only the curve infinitely near to ℓ_1 fits in the first case. On the other hand, σ contracts V and exceptional surfaces of blow ups. But \mathcal{Q} intersects V in the base locus of \mathcal{X} and if an exceptional surface is contracted, \mathcal{X} has a base curve in it. And \mathcal{Q} contains the base curves of \mathcal{X} , the only

exception happening when $\ell_1 \subset C_6$. Therefore the image of the base curve of \mathcal{Q} infinitely near to ℓ_1 is the base locus of \mathcal{Q}' .

Blowing up p , the curves \mathcal{Q}_p have multiplicity two in $(\ell_2)_p$ and $(\ell_3)_p$ and multiplicity one in $(\ell_1)_p$. Then \mathcal{Q}_p is the union of the line $(\Gamma_1)_p$ and movable conics through these three points. In particular, \mathcal{Q}_p has multiplicity three in p and it intersects a general plane through ℓ_1 in ℓ_1 and cubics with a double point in p .

Now blow up ℓ_1 . The intersection of \mathcal{Q}_1 with the exceptional divisor $E_{\ell_1} \cong \mathbb{F}_0$ is a curve of type $(1, 1)$. Since $\ell_1 \not\subset C_6$, this curve contains the three points $C_6 \cap E_{\ell_1}$. But fixed three non collinear points in \mathbb{F}_0 , there is only one curve of type $(1, 1)$ through them. And these points are indeed non collinear, since they lie on V_{ℓ_1} , which is irreducible. We have thus shown that:

$$\mathcal{Q}_{\ell_1} = t_L^1 \equiv (1, 1)$$

where $t_L = t_L^1$ is a fixed curve. By Lemma 4.2.1, $\mathcal{X}_{\ell_1} \equiv (3, 4)$ and it has three double points in t_L . Then t_L is mapped to a line $L = L_1$, which is the base locus of \mathcal{Q}' . It contains the point x_p , image of $(E_p)_{\ell_1} \equiv (1, 0)$.

Suppose now that $\ell_1 \subset C_6$. We'll explain the case $C_6 = 2\ell_1 + C_4$, the other cases being similar. As explained in a remark after Lemma 4.3.1, \mathcal{X} has a double curve u infinitely near to ℓ_1 , whereas \mathcal{Q} has multiplicity two along ℓ_1 and does not contain u . Then blowing up p gives:

$$\mathcal{Q} = t_1 + t_2 + t_3$$

keeping the notation of (4.3). A general plane through ℓ_1 intersects \mathcal{Q} in $2\ell_1$ and conics through p . Then the blow up in ℓ_1 gives $\mathcal{Q}_{\ell_1} \equiv (1, 2)$. The movable part of \mathcal{X}_{ℓ_1} , consisting of fibers, contracts E_{ℓ_1} to the line L_1 defined in Lemma 4.2.2. Then it follows that this is a double line of \mathcal{Q}' .

As remarked after Lemma 4.2.2, if $C_6 = 4\ell_1 + C_2$ or $C_6 = 6\ell_1$, \mathcal{X} maps curves infinitely near to ℓ_1 to singular curves infinitely near to L_1 . In these cases, these curves lie on the base locus of \mathcal{Q}' .

Since the base locus of \mathcal{Q} is contained in the base locus of \mathcal{X} , the base locus of \mathcal{Q}' is L_1 and possibly other curves infinitely near to L_1 . □

Remark 4.3.3. The line L_i in X was defined in two different ways in this section. In the above Lemma, it is defined as the base locus of \mathcal{Q}_i . In Lemma 4.2.2, L_i is defined as the contraction of E_{ℓ_i} when $\ell_i \subset C_6$. In the proof of Lemma 4.3.2 it is explained that these definitions coincide.

We now explain the first consequence of the results of this section. Unlike Chapter 3, different quartic fundamental surfaces produce different Bronowski threefolds, instead of representing different tangential projections of the same variety.

Proposition 4.3.4. *Let x and y be two points of the Bronowski threefold X such that the tangential projections τ_x and τ_y are birational. If the fundamental surfaces associated to τ_x and τ_y have degree four, then they are projectively equivalent.*

Proof. The projective equivalence classes of these quartic surfaces are given by (i), (ii) and (iii) in Proposition 4.1.1. In (i), V is the Steiner's Roman Surface and by Lemma 4.3.1, there are three pencils of quartic surfaces in X . Each of these pencils covers X and each quartic surface spans a \mathbb{P}^4 . In the same result, it is shown that in case (ii) there are two such pencils in X and in case (i) there is one. The goal is to show that in cases (ii) and (iii), the variety X does not contain another pencil covering X consisting of quartic surfaces, each spanning a \mathbb{P}^4 .

Note that there is no quartic surface belonging to two different families. This follows from the fact that a quartic surface (in \mathbb{P}^3) defines a unique complete linear system containing it (in this case, a pencil). Then we only need to show that there is no other quartic surface spanning a \mathbb{P}^4 through the point x from which is made the tangential projection. This shows that the cases (i), (ii) and (iii) produce different threefolds.

Suppose that, except from the quartic surfaces described in Lemma 4.3.1, there is another quartic surface S through x spanning a \mathbb{P}^4 . Since $S \subset X$, $T_x S$ is a subspace of $T_x X \cong \mathbb{P}^3$, the center of the projection. Therefore, every hyperplane containing $T_x X$ contains $T_x S$, the induced map on S is defined by a sublinear system of its tangential projection at x .

If S is smooth at x , the image of S is either a line ℓ or a base point of \mathcal{X} .

In the first case, a tangent hyperplane section of S at x (a quartic curve with a double point in x) is contracted to a point in ℓ . Therefore, \mathcal{X} has multiplicity four along ℓ , since σ maps points of ℓ to quartic curves. This implies that ℓ is a double line of V . By Proposition 4.2.1 and Lemma 4.2.2, S must be one of the surfaces described in these results.

Suppose now it is a base point of \mathcal{X} . But, as seen in Section 4.2 and in Proposition 4.1.4, a point of a line ℓ_i is not mapped to a surface. And neither is a point of C_6 , since these are mapped to points or curves in X .

If S is singular at x , the image of S is again a base point of \mathcal{X} . As seen above, this is not possible

Then, in both cases, S is a surface in \mathcal{Q}'_i .

□

This result also follows from the considerations in the next Section on the singularities of X . By Lemma 4.2.1 if $\ell_1 \prec \ell_2$, then ℓ_1 is mapped to a double line R_1 of X , and if $\ell_1 \prec \ell_2 \prec \ell_3$, then X has a second double line R_2 infinitely near to R_1 . In the next Section, we'll see this is the only way to produce double lines in X .

4.4 Singularities of X

According to Lemma 4.1.4, the point p is mapped to a point x_p of multiplicity four in X . Moreover, Proposition 4.2.1 asserts that if V is not the Steiner's Roman Surface, then X has one or more double lines R_i . Apart from these, the other singularities of X will depend, as in the other chapters, on the singularities of C_6 :

Lemma 4.4.1. *Let $x_q \neq x_p$ be an isolated singularity of X or a general point on a singular curve of X . Then x_q is either a point of a double line R_i , image of $\ell_i \in V$, or it is mapped by τ to a singular point of C_6 . This includes general points in non reduced components of C_6 .*

Proof. Remember first that a double line R_i is the contraction of E_{ℓ_i} , when V has a double line infinitely near to ℓ_i , as it was defined in Lemma 4.2.1.

As noted in the other chapters, x_q is mapped to a point q in the base locus of \mathcal{X} .

If $x_q \in \tau^{-1}(p)$, then it lies on the \mathbb{P}^4 spanned by $T_x X$ and x_p . Varying x in X , we find that x_q cannot be an isolated singular point, since by Terracini's Lemma two general tangent spaces of X are disjoint and $x_q \neq x_p$. By the same reason, if a singular curve of X contains x_q , it does not lie on $\tau^{-1}(p)$. Therefore it is projected to a curve in the base locus of \mathcal{X} .

This shows that if x_q is an isolated singular point or a general point in a singular curve of X , then $q \neq p$.

Suppose $x_q \notin R_i$. We'll prove that if q is not a singular point of C_6 , then x_q is a smooth point of X . This will be done by showing that a general tangent hyperplane section of X at x through x_q is smooth on x_q . Since x_q is either an isolated singular point or a general point in a singular line, this hyperplane section is mapped by τ to a general plane through q . This is the same technique used in Lemma 3.2.4.

Therefore, let Ω be a general plane through q . If $q \in \ell_j \setminus C_6$, then $\mathcal{X} \cap \Omega$ consists of degree 9 curves having multiplicity four in q and in two

other points and having multiplicity two in six points $C_6 \cap \Omega$. None of these points are infinitely near to q , since $x_q \notin R_j$ and $q \notin C_6$. Since the image of Ω via σ has degree nine, it follows that $\mathcal{X} \cap \Omega$ has no other base points. Then $x_q \in \tau^{-1}(q)$ is a smooth point of Ω .

If q is a smooth point of C_6 not lying on any of the double lines, then $\mathcal{X} \cap \Omega$ has three points of multiplicity four, has multiplicity two in q and in other five points. Since $q \notin \ell_j$ and since it is a smooth point of C_6 , none of the other base points is infinitely near to it. Then again the image of Ω is smooth in x_q .

Suppose then that q is a smooth point of C_6 lying on a double line ℓ_j of V (in particular, $\ell_j \not\subset C_6$). Then $\mathcal{X} \cap \Omega$ consists of degree nine curves having multiplicity four in $q = \ell_j \cap \Omega$ and in two other points not infinitely near to q , and having multiplicity two in six points of $C_6 \cap \Omega$. Since $q \in C_6$, one of the six double points q' lies infinitely near to q . Since q is a smooth point of C_6 , $\mathcal{X} \cap \Omega$ has no other double points infinitely near to q or q' .

Blowing up q , the exceptional curve $\Omega_q \subset \Omega$ is mapped by σ to a conic. Blowing up q' , this second exceptional curve is also mapped to a conic. Then none of these curves is contracted. Moreover, σ is an isomorphism outside V , so the only curve in Ω through q that is contracted is $V \cap \Omega$, which is mapped to the smooth point x . Hence x_q is a smooth point in the image of Ω . This completes the proof. \square

The simplest singularities of X are the images of singular points of C_6 not lying on the double lines of V :

Lemma 4.4.2. *Let q be a singular point of C_6 not lying on any ℓ_i . Then q is mapped to a double point of X and the tangent cone of X in this point has:*

- rank 4, if q is the transversal intersection of two simple branches of C_6 ;
- rank 3, if q is a cuspidal point, a point of contact of two simple branches or a general point of a double component of C_6 ;
- rank 2, that is, it is a pair of three-dimensional planes, if q is the intersection of three simple branches, the intersection of a cuspidal and a simple branch or the intersection of two cuspidal branches of C_6 .

Proof. The curve C_6 is the image via $\bar{\tau}$ of a plane cubic and this map is an isomorphism out of the three double lines of V . But in Chapter 2, the curve C_4 in the base locus of \mathcal{X} is also the image via $\bar{\tau}$ of a cubic. And it is also a double curve of \mathcal{X} . Moreover, the singularities of C_4 do not lie on the two simple lines of \mathcal{X} (see Lemma 2.1.1), since the tangential projection is general. Then the two maps $\bar{\tau}$ coincide on the point q .

Therefore the singularities produced by points of C_6 not lying on any of the double lines of V are the same as the ones produced by the curve C_4 in Chapter 2. Then the result follows directly from Lemma 2.2.12. \square

As described in Section 4.1, the map $\bar{\tau}$ restricts to a double cover of each proper double line ℓ_i of V . This double cover is ramified at two points, namely q_j^i , for $j = 1, 2$.

Before studying the singularities of C_6 lying on a line ℓ_i , we'll describe the behaviour of \mathcal{X} under the blow up at a general point of ℓ_i . To simplify the notation, this will be done for $i = 1$.

Lemma 4.4.3. *Suppose ℓ_1 is a proper line of V , let q be a point of ℓ_1 distinct from p and let $\{\hat{q}_1, \hat{q}_2\}$ be the preimages of q via $\bar{\tau}$.*

If V does not have a double line infinitely near to ℓ_1 , then blowing up q gives $V_q = u_1 + u_2$, a pair of lines through $(\ell_1)_q$. Moreover \mathcal{X}_q is a linear system of quadruples of lines through $u_1 \cap u_2$. If $\mathcal{X}_q = 2u_1 + 2u_2$, then E_q is contracted to a point in L_1 .

If ℓ_2 is infinitely near to ℓ_1 and ℓ_3 is a proper line of V , then blowing up q gives $\mathcal{X}_q = 4t$, where $t = (\Gamma_3)_q$. After blowing up ℓ_1 and ℓ_2 , the blow up at t gives, in $E_t \cong \mathbb{F}_0$, $\mathcal{X}_t \equiv (0, 4)$ and $V_t = u_1 + u_2 \equiv (0, 2)$. If $\mathcal{X}_t = 2u_1 + 2u_2$, then E_t is contracted to a point in R_1 .

If $\ell_1 \prec \ell_2 \prec \ell_3$, then again blowing up q gives $\mathcal{X}_q = 4t'$, where $t' = (\Gamma_3)_q$. After blowing up ℓ_1, ℓ_2, ℓ_3 and t' , V and \mathcal{X} intersect $E_{t'}$ in a curve t with multiplicity two and four respectively. The blow up at t gives, in $E_t \cong \mathbb{F}_0$, $\mathcal{X}_t \equiv (0, 4)$ and $V_t = u_1 + u_2 \equiv (0, 2)$. If $\mathcal{X}_t = 2u_1 + 2u_2$, then E_t is contracted to a point in R_1 .

In the three cases, the lines u_1 and u_2 are infinitely near if and only if q is q_1^1 or q_2^1 . Moreover, the indexes can be chosen to satisfy the following property: If a curve in $E = \bar{\tau}^{-1}(V)$ contains \hat{q}_j , for $j \in \{1, 2\}$, then its image in V intersects E_t in a point of u_j .

Proof. Suppose first that V does not have a double line infinitely near to ℓ_1 . By Lemma 4.1.3, the tangent cone of V on q is a pair of planes containing ℓ_1 ,

which coincide if q is q_1^1 or q_2^1 . Then $V_q = u_1 + u_2$. Since \mathcal{X} has multiplicity four in q and along ℓ_1 , \mathcal{X}_q is the union of four lines through $(\ell_1)_q = u_1 \cap u_2$.

Remember that the line L_1 is the base locus of \mathcal{Q}'_1 . Suppose $\mathcal{X}_q = 2u_1 + 2u_2$, so that E_q is mapped to a point. This implies that $q \in C_6$ and $(C_6)_q$ consists of points in both u_1 and u_2 . Then \mathcal{Q}_q has three non collinear base points, that is, q is a singular point of \mathcal{Q} . Now blow up ℓ_1 . As shown in the proof of Lemma 4.3.2, $\mathcal{Q}_{\ell_1} = t_L^1$ is a fixed curve of type $(1, 1)$ and it is mapped to L_1 . Since \mathcal{Q} is singular on q , t_L^1 contains $E_q \cap E_{\ell_1} \equiv (1, 0)$. Therefore the contraction of E_q lies on L_1 . This proves the first part.

Suppose now that ℓ_2 is infinitely near to ℓ_1 and that ℓ_3 is a proper line of V . Start blowing up q . Then \mathcal{X}_q has multiplicity four in $(\ell_1)_q$ and in $(\ell_2)_q$, which is infinitely near to $(\ell_1)_q$ in the direction of $t = (\Gamma_3)_q$. Therefore $\mathcal{X}_q = 4t$. For the same reason, $V_q = 2t$.

Blow up ℓ_1 and $\ell_2 = \Gamma_3 \cap E_{\ell_1}$. The self-intersection of $t = \Gamma_3 \cap E_q$ in E_q has decreased from 1 to -1 with the blow ups of ℓ_1 and ℓ_2 . In Γ_3 , t is the exceptional curve of the blow up at q , giving $t^2 = -1$. Since ℓ_1 and ℓ_2 lied in Γ_3 , this is not affected by their blow ups. Then the normal bundle of t is:

$$N_t = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

Therefore, blowing up t gives $V_t \equiv (0, 2)$ and $\mathcal{X}_t \equiv (0, 4)$. Set $V_t = u_1 + u_2$.

As explained in Lemma 4.2.1, the moving part of \mathcal{X}_{ℓ_1} consists of moving fibers, which map E_{ℓ_1} to the line R_1 . If $\mathcal{X}_t = 2u_1 + 2u_2$, then E_t is contracted to a point, which coincides with the image of E_q . Since E_q intersects E_{ℓ_1} in a fiber, it follows that the image of E_t lies on R_1 . This completes the proof of the second part.

Now suppose that $\ell_1 \prec \ell_2 \prec \ell_3$. This case is illustrated in Figure 4.4.

Blow up q , giving again $\mathcal{X}_q = 4t'$ and $V_q = 2t'$, with the new notation $t' = (\Gamma_3)_q$. Consider the blow ups along ℓ_1 , ℓ_2 and ℓ_3 .

After this, t' is no longer the complete intersection of E_q with Γ_3 , but $E_q \cap V = 2t'$. In E_q , t' is a smooth curve with $(t')^2 = -2$. In V , t' is a double curve. But in the normalization of V , this double curve is the union of two disjoint smooth curves, the exceptional curves of the blow up at two smooth points. Therefore, $(t')^2 = -1$ and its normal bundle is:

$$N_{t'} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

Blow up t' . In $E_{t'} \cong \mathbb{F}_1$, clearly $V_{t'} = 2t \equiv 2e_1$. On the other hand, \mathcal{X} had no moving intersection with E_q , so $\mathcal{X}_{t'} = 4t$. The normal bundle of t is:

$$N_t = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

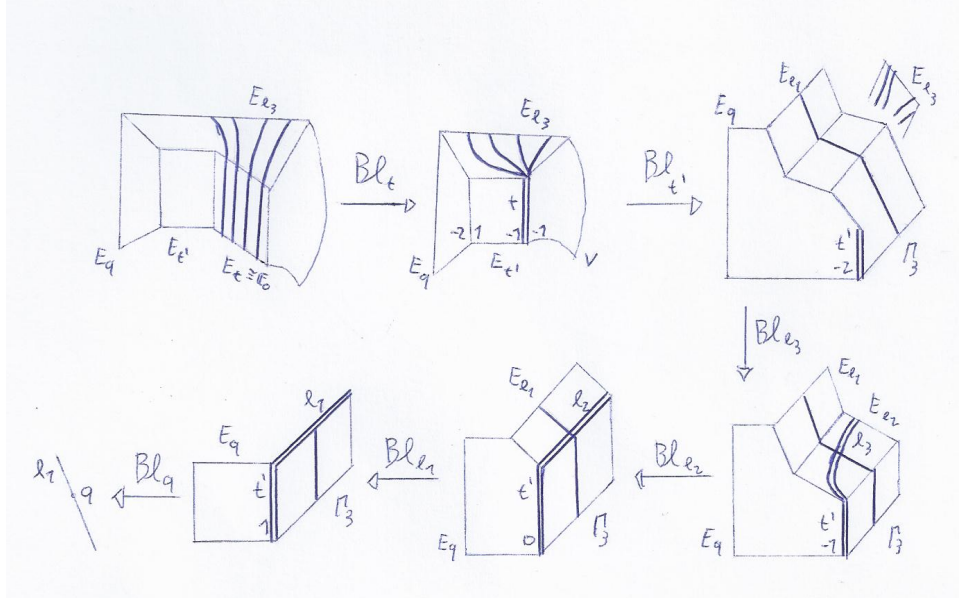


Figure 4.4: The blow up of \mathcal{X} when $\ell_1 \prec \ell_2 \prec \ell_3$

Hence, the blow up at t gives $V_t \equiv (0, 2)$ and $\mathcal{X}_t \equiv (0, 4)$, as stated. If $\mathcal{X}_t = 2u_1 + 2u_2$, the surfaces E_t , $E_{t'}$ and E_q are contracted to the same point. Since E_q intersects E_{ℓ_1} in a line, this point lies on R_1 . The third part is proved.

Now, let $\hat{C} \subset E$ be a line through \hat{q}_1 , let $C \subset V$ be its image via $\bar{\tau}$ and let Σ be the plane containing C . Then $\Sigma \cap V = C + C'$, where C' is a conic through q . Following the blow ups made above, we get, in the two last cases, $\Sigma_t \equiv (1, 0)$. It intersects $V_t = u_1 + u_2 \equiv (0, 2)$ in two points. The same happens in the first case, where Σ_q is a line. Then one point, say $\Sigma \cap u_1$, is the intersection with C . The other point is the intersection with C' .

By continuity, the lines through \hat{q}_j are mapped to conics intersecting u_j , for $j \in \{1, 2\}$. Taking tangent lines (or tangent cones) of curves in E , the same result follows for curves through \hat{q}_j .

The lines u_1 and u_2 are infinitely near if and only if every plane Σ as above intersects V_t (or V_q) in two infinitely near points. By the above result, this is equivalent to \hat{q}_1 being infinitely near to \hat{q}_2 , that is, q is q_1^1 or q_2^1 . \square

We now consider the singularities of X coming from a singular point of

C_6 lying on $\ell_i \setminus p$. The study of these singularities is very similar to the singularities described in Chapter 3 which lied in the double line s of V .

Lemma 4.4.4. *Let $q \neq p$ be a singular point of C_6 lying on ℓ_i , a proper double line of V . Let \hat{C} be the strict transform of C_6 and $\{\hat{q}_1, \hat{q}_2\}$ be the preimages of q via $\bar{\tau}$.*

If \hat{C} contains both \hat{q}_1 and \hat{q}_2 , then X has a triple point that is mapped to q . This point lies on R_i if V has a double line infinitely near to ℓ_i ; otherwise it lies on L_i .

If \hat{C} contains only one point among $\{\hat{q}_1, \hat{q}_2\}$, then X has a double point that is projected to q .

These considerations are also valid if $q = q_j^i$, in which case \hat{q}_1 and \hat{q}_2 are infinitely near.

Proof. Set $i = 1$ and $\ell = \ell_1$. Suppose that q is not q_j^1 , so that \hat{q}_1 and \hat{q}_2 are not infinitely near. If $q = q_j^i$, the same argument can be used, noting that \hat{C} is tangent to $\hat{\ell}$ on $\hat{q} = \bar{\tau}^{-1}(q)$.

Suppose first that V does not have a double line infinitely near to ℓ_1 .

By Lemma 4.4.3, blowing up q gives $V_q = u_1 + u_2$ and \mathcal{X}_q is the union of four lines through $u_1 \cap u_2$. If \hat{C} contains \hat{q}_1 and \hat{q}_2 , the same Lemma asserts that C_6 intersects E_q in points of both u_1 and u_2 , which implies that $\mathcal{X}_q = 2u_1 + 2u_2$. If \hat{C} contains only \hat{q}_1 , then C_6 intersects E_q in points of u_1 and $\mathcal{X}_q = 2u_1 + \{\text{pairs of lines}\}$.

Now repeat the arguments in the proof of Proposition 3.2.9, of Chapter 3: in the first case E_q is mapped to a triple point of X ; in the second case E_q is mapped to a curve and u_1 is mapped to a double point of X lying on this curve.

By Lemma 4.4.3, the triple point lies on R_1 or L_1 , as stated.

Suppose now that V has a double line ℓ_2 infinitely near to ℓ_1 . The line ℓ_3 can be either proper or infinitely near to ℓ_2 . In Lemma 4.4.3, both cases are considered: If ℓ_3 is a proper line of V , then blowing up q , ℓ_1 , ℓ_2 and a curve $t \in E_q$, gives, in $E_t \cong \mathbb{F}_0$, $\mathcal{X}_t \equiv (0, 4)$ and $V_t = u_1 + u_2 \equiv (0, 2)$. If ℓ_3 is infinitely near to ℓ_2 , then blowing up q , ℓ_1 , ℓ_2 , a curve $t' \in E_q$ and a curve $t \in E_{t'}$, gives, in $E_t \cong \mathbb{F}_0$, $\mathcal{X}_t \equiv (0, 4)$ and $V_t = u_1 + u_2 \equiv (0, 2)$.

So in both cases we have:

$$V_t = u_1 + u_2 \equiv (0, 2)$$

and $\mathcal{X}_t \equiv (0, 4)$, where t is a curve infinitely near to q . According again to Lemma 4.4.3, if a curve in $E = \bar{\tau}^{-1}(V)$ contains \hat{q}_j , for $j \in \{1, 2\}$, then its image in V intersects E_t in a point of u_j .

Therefore, if \hat{C} contains both \hat{q}_1 and \hat{q}_2 , then:

$$\mathcal{X}_t = 2u_1 + 2u_2$$

and E_t is contracted to a point. It lies on R_1 , by Lemma 4.4.3. Since t is infinitely near to E_q , this point is projected by τ to q .

To prove that it is a triple point of X , use Lemma 1.3.3. Let Ω be a general plane through q . Making the blow ups described in Lemma 4.4.3, the line t intersects Ω in one point. Blowing up this point, $\mathcal{X} \cap \Omega$ intersects the exceptional curve $e = E_t \cap \Omega$ in two fixed double points. After the blow up at these two points, we get $e^2 = -3$ in Ω and it is contracted to a triple point of $\sigma(\Omega)$. Then by Lemma 1.3.3, it is a triple point of X .

If, on the other hand, \hat{C} contains only one point among $\{\hat{q}_1, \hat{q}_2\}$, say \hat{q}_1 , then:

$$\mathcal{X}_t = 2u_1 + \{\text{pairs of lines}\}$$

where the moving part is of type $(0, 2)$. Then E_t is mapped to a curve. The line $u = u_1$ has normal bundle:

$$N_u = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

The blow up at u gives $\mathcal{X}_u \equiv 2e_1 + 2f_1$, having two or more double points in $(C_6)_u$. Therefore, \mathcal{X}_u is a fixed curve and E_u is mapped to a point.

To prove that it is a double point of X , let Ω be a general plane through q . Perform the blow ups described in Lemma 4.4.3 and the blow ups of t and u . Then $\mathcal{X} \cap \Omega$ intersects $e = E_u \cap \Omega$ in one fixed double point. After the blow up at this point, we get $e^2 = -2$ in Ω . Then e is mapped to a double point of $\sigma(\Omega)$ and lemma 1.3.3 concludes the argument. \square

Next, the case in which C_6 contains a non reduced component is analysed. If this component is a line, this has been studied in Section 4.2. Since C_6 is the image via $\bar{\tau}$ of a cubic, the other possibilities are that a non reduced component is a double conic or a triple conic.

Lemma 4.4.5. *Suppose C is a double conic in C_6 , that is, $C = C_1 \prec C_2$ are double curves of \mathcal{X} . Then C is mapped to a double conic of X and C_2 is mapped to a Veronese quartic surface.*

If $C_6 = 3C$, that is, $C = C_1 \prec C_2 \prec C_3$ are contained in C_6 , then C_3 is mapped to a Veronese quartic surface through x and C_1 and C_2 are mapped to infinitely near double conics of X lying on this surface.

Moreover, if for some $i \in \{1, 2, 3\}$, $q = C \cap \ell_i$ is mapped to a triple point of X , then this point lies on the double conic, unless $C_6 \neq 3C$ and q is q_1^i or q_2^i .

Proof. Let Σ be the plane containing C . Suppose first that C does not contain p and let Q be the cone over C with vertex p .

The linear system \mathcal{X} intersects Σ in $2C$ and degree five curves, which intersect C in ten points. Among these, three double points of the moving part of $\mathcal{X} \cap \Sigma$ lie on the lines ℓ_i . On the other hand, V intersects Σ in C and another conic, intersecting C in three points in the lines ℓ_i and one other point.

The intersection of \mathcal{X} with Q is $2C + 4\ell_1 + 4\ell_2 + 4\ell_3$ and moving conics. These conics do not contain p , since the fixed part has already multiplicity twelve in this point and the tangent cone of \mathcal{X} in p is the union of the planes Γ_i with multiplicity two. Therefore, the moving part of $\mathcal{X} \cap Q$ intersects C in two points. The surface V cuts Q in $C + 2\ell_1 + 2\ell_2 + 2\ell_3$.

First blow up the lines ℓ_1, ℓ_2, ℓ_3 . After that, $C = \Sigma \cap Q$ has normal bundle:

$$N_C = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$$

Blowing up C gives $\Sigma_C \equiv e_1 + f_1$ and $Q_C \equiv e_1$. Then $C_2 = V_C \equiv e_1 + f_1$, which is, by hypothesis, a double curve of \mathcal{X}_C . Therefore:

$$\mathcal{X}_C \equiv 2e_1 + 4f_1 \equiv 2C_2 + \{2f_1\}$$

The moving part maps E_C to a conic in X . The curve $C_2 = V \cap E_C$ has normal bundle:

$$N_{C_2} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

Blowing up C_2 gives $\mathcal{X}_{C_2} \equiv (2, 2)$ and $V_{C_2} \equiv (0, 1)$. If C is a double curve in C_6 , \mathcal{X}_{C_2} has one fixed double point, corresponding to the intersection with the other component of C_6 , a conic. Then \mathcal{X}_{C_2} is birationally equivalent to a linear system of conics in \mathbb{P}^2 with no base points, and E_{C_2} is mapped to a quartic Veronese surface. It contains x , the image of V_{C_2} .

If C is a triple curve in C_6 , $\mathcal{X}_{C_2} \equiv 2C_3 + (2, 0)$, where $C_3 = V_{C_2}$. Then E_{C_2} is mapped to a conic. Blowing up $C_3 = E_{C_2} \cap V$ gives $E_{C_3} \cong \mathbb{F}_1$, with $V_{C_3} \equiv e_1$. Then $\mathcal{X}_{C_3} \equiv 2e_1 + 2f_1$ with no base points and again E_{C_3} is mapped to a Veronese surface through x .

Now suppose C contains p . The case $C_6 = 2C + C'$, with C' intersecting ℓ_1 in one point, is illustrated in Figure 4.5.

The intersection $\Sigma \cap V$ consists of C and a pair of lines through p . Since the only lines in V are its double lines, it follows that $\Sigma \cap V = C + 2\ell_j$, for

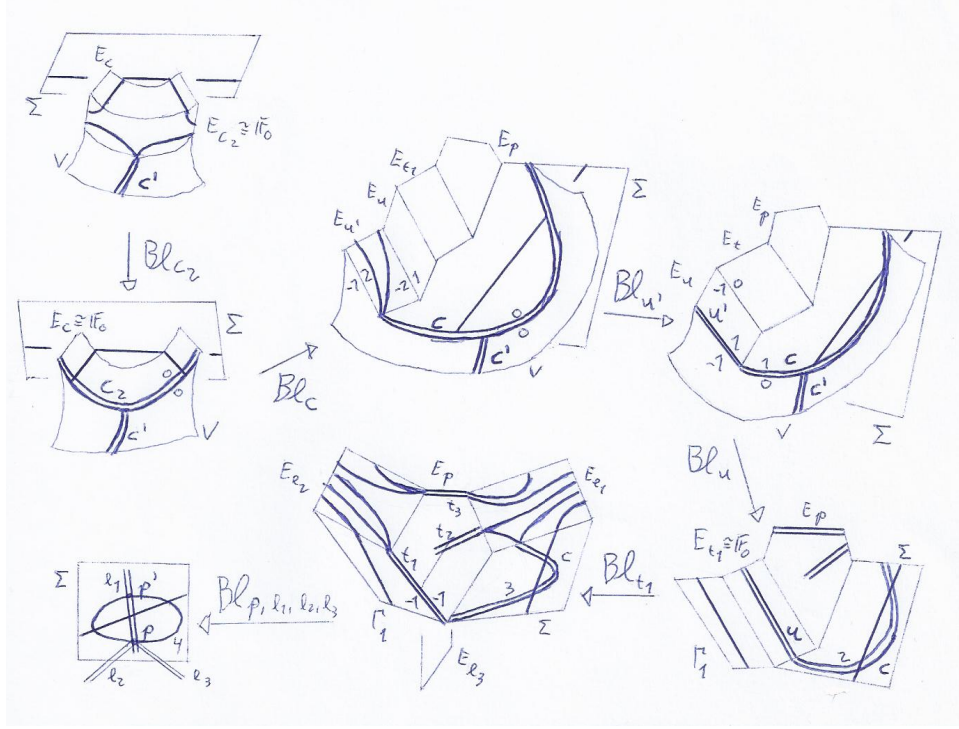


Figure 4.5: Blow ups when $C_6 = 2C + C'$, with C containing p

some j . For simplicity, set $j = 1$. This implies that C cuts ℓ_1 in a point p' other than p , but cuts ℓ_2 and ℓ_3 only in p . Then $C_2 \cap \Gamma_1 = 2p$, and $C_2 \cap \Gamma_i = p + p'$, for $i \in \{2, 3\}$. Note that $p' \neq p$, otherwise Σ would be a component of the tangent cone of V in p , so C would be a double line of V .

Moreover:

$$\mathcal{X} \cap \Sigma = 2C + 4\ell_1 + \{\text{lines}\}$$

Start blowing up p , giving, as explained in (4.5):

$$\mathcal{X}_p = 2t_1 + 2t_2 + 2t_3$$

where $t_i = (\Gamma_i)_p$. Then C_p is a point in t_1 and Σ_p is the line described by this point and $(\ell_1)_p$.

Next blow up the three lines ℓ_1 , ℓ_2 and ℓ_3 . Before blowing up C , we'll blow up t_1 , in order to show there are base curves of \mathcal{X} infinitely near to it. The normal bundle of $t_1 = (\Gamma_1)_p$, after these blow ups, is:

$$N_{t_1} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

Therefore, blowing up t_1 , gives, in $E_{t_1} \cong \mathbb{F}_0$, $\mathcal{X}_{t_1} = 2u \equiv (0, 2)$, since C_{t_1} is a double point. The normal bundle of $u = E_{t_1} \cap V$ is:

$$N_u = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

Then blowing up u gives $V_u \equiv e_1 + f_1$ and $\mathcal{X}_u \equiv 2e_1 + 2f_1$. Again, C_u is a double point of \mathcal{X}_u . But there is a second double point, namely $(C_2)_u$, which is infinitely near to C_u . Both points lie on $u' = V_u$, so $\mathcal{X}_u = 2u'$.

We proceed blowing up $u' = E_u \cap V$, which gives $E_{u'} \cong \mathbb{F}_2$ and $\mathcal{X}_{u'} \equiv 2e_2 + 4f_2$. If C is a triple curve in C_6 , then $\mathcal{X}_{u'}$ has three double points: $C_{u'} \prec (C_2)_{u'} \prec (C_3)_{u'}$. All of these points lie on $u'' = V_{u'} \equiv e_2 + 2f_2$, giving $\mathcal{X}_{u'} = 2u''$. The same happens if $C_6 = 2C + C'$, with C' intersecting ℓ_1 in p and in a second point, giving a third double point of $\mathcal{X}_{u'}$. In both cases, similar considerations show that there are no further base curves of \mathcal{X} infinitely near to u'' .

On the other hand, if $C_6 = 2C + C' \neq 3C$, with C' not as above, then $\mathcal{X}_{u'}$ intersects $V_{u'}$ in two double points, so it's not a fixed curve. This is the case illustrated in Figure 4.5.

We now concentrate on $C = \Sigma \cap V$. The following considerations hold for $C_6 = 2C + C'$ and for $C_6 = 3C$. In Σ , the blow up of p , t_1 , u and u' decreased the self-intersection of C , giving $C^2 = 0$. In V (blown up in p), we have $C^2 = 0$. Then the normal bundle of C is:

$$N_C = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

Blow up C , giving $C_2 = V_C \equiv (0, 1)$. Since $\mathcal{X} \cap \Sigma$ intersects E_C in two moving points, it follows that:

$$\mathcal{X}_C = 2C_2 + \{\text{moving part}\} \equiv (0, 2) + \{(2, 0)\}$$

and again the moving part maps E_C to a conic.

The curve $C_2 = E_C \cap V$ has normal bundle:

$$N_{C_2} = \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(0)$$

Blow up C_2 . Then $\mathcal{X}_{C_2} \equiv (2, 2)$ and $V_{C_2} \equiv (0, 1)$.

To study the base points of \mathcal{X}_{C_2} , suppose first $C_6 = 2C + C'$, with $C' \neq C$. If C' intersects ℓ_1 in two points, then $(u'')_{C_2}$ is a double point of \mathcal{X}_{C_2} . If C' intersects ℓ_1 only in p , then it intersects Σ (before the blow ups) in p and in a point lying on C . Therefore, C' intersects E_{C_2} , and this is a double point of \mathcal{X}_{C_2} . Finally, if C' does not contain p , then again $(C')_{C_2}$ is a double point of \mathcal{X}_{C_2} .

Then in all cases $\mathcal{X}_{C_2} \equiv (2, 2)$ has one double point. By Lemma 1.1.1, this linear system corresponds, in \mathbb{P}^2 , to curves of degree four having three fixed double points. Applying a standard quadratic map gives a base-point free linear system of conics. Hence E_{C_2} is mapped to a Veronese surface.

Now suppose $C_6 = 3C$. Then:

$$\mathcal{X}_{C_2} = 2C_3 + \{\text{moving part}\} \equiv (0, 2) + \{(2, 0)\}$$

Then, blowing up $C_3 = E_{C_2} \cap V$, gives $E_{C_3} \cong \mathbb{F}_0$ and $V_{C_3} \equiv (0, 1)$. The linear system $\mathcal{X}_{C_3} \equiv (2, 2)$ has one double point in $(u'')_{C_3}$. As before, it maps E_{C_3} to a quartic Veronese surface.

The next step is to show that X has multiplicity two in the conics. Let q be a general point in C and let Ω be a general plane through q . Then q is a double point of $\mathcal{X} \cap \Omega$, which has another double point infinitely near to it. If $C_6 = 3C$, then there is a third infinitely near double point.

Blowing up the double points, we get one curve (or two, if $C_6 = 3C$) with self-intersection (-2) in Ω having no intersection with $\mathcal{X} \cap \Omega$. Then $\sigma(\Omega)$ has multiplicity two in $\sigma(q)$ and another double point infinitely near to it when $C_6 = 3C$. Then Lemma 1.3.3 completes the proof.

To prove the last part, set $\ell = \ell_i$ and suppose that $q = C \cap \ell$ is mapped to a triple point of X . According to Lemma 4.4.4, E_q is contracted to a point. Then, after the blow up at ℓ , the fiber f^q over q in E_ℓ , which contains the point C_ℓ , is contracted to a triple point. Also note that:

$$V_\ell \equiv (2, 2) \equiv f^p + (1, 2)$$

where f^p is the fiber over p . The curve of type $(1, 2)$ is reducible if and only if V has a double line infinitely near to ℓ . In this case, blow up this line in order to find an irreducible curve of type $(1, 2)$ in the other exceptional divisor. Then V_ℓ is tangent to f^q if and only if q is q_1^i or q_2^i . In this case, the point of tangency is C_ℓ .

Suppose first that q is q_1^i or q_2^i and $C_6 \neq 3C$. Blowing up C , f^q and V_ℓ intersect E_C in the same point of $C_2 = V_C$. Blowing up C_2 , E_{C_2} is mapped to a quartic surface and E_C is contracted to a conic in it. In this stage, $f^q \subset E_\ell$ does not intersect either V_ℓ or E_C . Then it is mapped to a point not lying on the image of E_C . Therefore the image of q does not lie on the double conic.

Now suppose that $C_6 = 3C$. Then, after the blow ups of $\ell_1, \ell_2, \ell_3, C$ and C_2 , f^q does not intersect V_ℓ , even if q is q_1^i or q_2^i . Moreover, as seen before:

$$\mathcal{X}_{C_2} \equiv (2, 2) \equiv 2C_3 + (2, 0)$$

and the point $f^q \cap E_{C_2}$ does not lie on $C_3 = V_{C_2}$. Hence f^q is mapped to a point in the image of C_2 , which is a conic infinitely near to the image of C . Hence it is a point in the double conic.

Finally, suppose that q is not q_1^i nor q_2^i and $C_6 \neq 3C$. Then, after the blow up at ℓ , V_ℓ intersects f^q transversally in C_ℓ . Blowing up C , f^q does not intersect V_ℓ any more. Then Blowing up $C_2 = V_C$, f^q still intersects E_C , which is mapped to the double conic containing the image of f^q . \square

The following proposition summarizes the results on singularities of X .

Proposition 4.4.6. *The threefold X has multiplicity four in x_p . If V has a double line infinitely near to ℓ_i , then X has a double line R_i , which does not contain x_p and is mapped to ℓ_i . The other singularities of X are projected to singularities of C_6 .*

Let q be an isolated singular point of C_6 , then it is the image of a singular point $x_q \in X$. If q does not lie on any of the double lines of V , then x_q is a double point of X . If q lies on ℓ_i and the strict transform of C_6 via $\bar{\tau}$ contains both preimages of q , then x_q is a triple point of X lying on R_i or L_i . If not, it is a double point.

Let C be a non reduced component of C_6 . If $C = \ell_i$, then X has a triple line L_i , which is projected to ℓ_i . If C is a conic, then it is mapped to a double conic of X .

Proof. These results follow from Proposition 4.2.1, Lemma 4.2.2, Lemma 4.4.1, Lemma 4.4.2 and Lemma 4.4.4. \square

We recall a remark made before Lemma 4.2.2. If, for instance, ℓ_2 is infinitely near to ℓ_1 and C_6 contains ℓ_1 , then \mathcal{X} has a double base curve infinitely near to ℓ_1 determined by V . But since \mathcal{X} has multiplicity four along ℓ_2 , this curve lies infinitely near to ℓ_2 . Then in this case we say that C_6 contains ℓ_2 , instead of ℓ_1 . This avoids confusion among the lines R_i and L_i .

4.5 Description of the general Bronowski threefold of degree 9

In this Section we investigate the Bronowski threefold in which the fundamental surface V is the Steiner's Roman Surface and in which C_6 is a

smooth curve not containing p and intersecting the lines ℓ_i transversally. By Proposition 4.4.6, the only singularity of X is x_p , a point of multiplicity four.

The following result is very similar to Proposition 3.2.17:

Proposition 4.5.1. *Let X be the general Bronowski threefold of degree 9. Then for $i \in \{1, 2, 3\}$, X lies on the cone with vertex L_i over a Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. In each of these cones, let H_0 be the class of a plane section and let H_1 be the class of a \mathbb{P}^4 of the ruling. Then:*

$$X \equiv 4H_0^2 - 3H_0H_1$$

Proof. For simplicity, set $i = 1$. By Remark 1.4.3, X is linearly normal. Then, by Theorem 1.4.4, X lies on a rational normal scroll F , described by the \mathbb{P}^4 's spanned by the quartics in \mathcal{Q}'_1 (see Lemma 4.3.2). These \mathbb{P}^4 's have in common the line $L_1 \in X$, so F is the cone with vertex L_1 over a scroll Y in \mathbb{P}^5 . It contains a one-dimensional family of disjoint planes, so Y is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$.

Since V is the Steiner's Roman surface, X lies on three distinct cones. The vertex of each of these cones is a line L_i . Since the cones are distinct and since each cone is the intersection of three quadric hypersurfaces, it follows that X lies on the intersection of F with two quadrics Q_1 and Q_2 . Note that $Q_1 \cap Q_2$ intersects a \mathbb{P}^4 of the ruling in a surface of \mathcal{Q}'_1 . By Lemma 4.3.2, this surface is a weak Del Pezzo quartic surface having a double point in $x_p \in L_1$.

Let $\eta : \text{Bl}_{L_1}(\mathbb{P}^7) \rightarrow \mathbb{P}^7$ be the blow up of L_1 , let G be the strict transform of F and E be the intersection of G with the exceptional divisor. Set H_0 and H_1 for the strict transforms in G of the classes H_0 and H_1 in F . Let Q'_1 and Q'_2 be the total transforms via η of $Q_1 \cap F$ and $Q_2 \cap F$, and let X' be the strict transform of $X \subset F$.

The divisors Q'_1 and Q'_2 contain both X' and E . So let S be residual intersection, that is:

$$Q'_1 \cap Q'_2 = X' + E + S$$

Since $Q_1 \cap Q_2$ intersects a \mathbb{P}^4 of the ruling of F in a weak Del Pezzo surface, S does not contain E . And since X intersects a \mathbb{P}^4 of the ruling of F in a quartic surface, it follows that S consists of threefolds contained in rulings of G . Moreover:

$$\deg(\eta(S)) = \deg(Q_1 \cap Q_2 \cap F) - \deg(X) = 12 - 9 = 3$$

These considerations imply that $S \equiv 3H_0H_1$. Therefore:

$$X' + E \equiv 4H_0^2 - 3H_0H_1$$

Taking the image via η in \mathbb{P}^7 , the result follows. □

This description gives us an important result:

Theorem 4.5.2. *The general Bronowski threefold of degree 9 is an OADP variety.*

Proof. By Proposition 4.5.1, $X \equiv 4H_0^2 - 3H_0H_1$ in a cone F over Y with vertex L , where Y is a Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. Following [CMR, Example 2.6], there is a smooth quadric surface $Q \subset Y$ not intersecting L such that if X intersects the \mathbb{P}^5 spanned by Q and L in an OADP surface, then X is an OADP variety. The quadric Q (and the \mathbb{P}^5 spanned by Q and L) depends on the choice of a general point in \mathbb{P}^7 , and X is OADP if there is a unique secant line to X through this point.

Since $Q \subset Y$ and $X \subset F$, the intersection of X with the \mathbb{P}^5 is a surface of type $4H_0^2 - 3H_0H_1$ in the cone F' over Q with vertex L , where now H_0 is the class of a section of F' and H_1 is the class of a \mathbb{P}^3 of the ruling. Then the degree of this intersection is:

$$(4H_0^2 - 3H_0H_1)H_0^2 = 8 - 3 + 0 = 5$$

Since X is non degenerate, this degree five surface also is. This follows from the fact that hyperplane sections of a non degenerate variety are non degenerate. The classification of non degenerate degree five surfaces in \mathbb{P}^5 is well known (see [Na] or [Cos]), giving one of the following:

- A (possibly weak) Del Pezzo surface
- A projection to \mathbb{P}^5 of a degree five scroll in \mathbb{P}^6 ;
- A cone over an elliptic curve of degree five in \mathbb{P}^4 .

The two last surfaces are ruled by lines. However, there is no line in X passing through a general point of it. Indeed, this would imply that the linear system $|H_{X,x}|$ has a base point, which is not the case. Therefore, since Q depends on a general point of \mathbb{P}^7 , it follows that the intersection of X with the \mathbb{P}^5 spanned by Q and L is a (possibly weak) Del Pezzo Surface of degree five. And this is an OADP surface (see [CMR]). Then X is an OADP variety. □

The other Bronowski threefolds of degree nine are degenerations of the general one studied in this Section. Then those are also OADP varieties. We will not give the details here.

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